

Identification of the theory of multi-dimensional orthogonal polynomials with the theory of symmetric interacting Fock spaces with finite dimensional 1-particle space

LUIGI ACCARDI
CENTRO VITO VOLTERRA
UNIVERSITÀ DI ROMA TOR VERGATA
VIA DI TOR VERGATA, 00133 ROMA, ITALY
E-MAIL: accardi@volterra.uniroma2.it

ABDESSATAR BARHOUMI
CARTHAGE UNIVERSITY, TUNISIA
NABEUL PREPARATORY ENGINEERING INSTITUTE
DEPARTMENT OF MATHEMATICS
CAMPUS UNIVERSITAIRE - MREZGUA - 8000 NABEUL
E-MAIL: abdessatar.barhoumi@ipein.rnu.tn

AMEUR DHAHRI
CENTRO VITO VOLTERRA
UNIVERSITÀ DI ROMA TOR VERGATA
VIA DI TOR VERGATA, 00133 ROMA, ITALY
E-MAIL: ameur@volterra.uniroma2.it

Abstract

The identification mentioned in the title allows a formulation of the multidimensional Favard Lemma different from the ones currently used in the literature and which exactly parallels the original 1-dimensional formulation in the sense that the positive Jacobi sequence is replaced by a sequence of positive Hermitean (square) matrices and the real Jacobi sequence by a sequence of Hermitean matrices of the same dimension. Moreover, in this identification, the multi-dimensional extension of the compatibility condition for the positive Jacobi sequence becomes the condition which guarantees the existence of the creator in an interacting Fock space. The above result opens the way to the program of a purely algebraic classification of probability measures on \mathbb{R}^d with finite moments of any order.

In this classification the usual Boson Fock space over \mathbb{C}^d is characterized by the fact that the positive Jacobi sequence is made up of identity matrices and the real Jacobi sequences are identically zero.

The quantum decomposition of classical real valued random variables with all moments is one of the main ingredients in the proof.

Keywords: Multidimensional orthogonal polynomials; Favard theorem; Interacting Fock space; Quantum decomposition of a classical random variable

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1 Introduction

The theory of orthogonal polynomials is one of the classical themes of calculus since almost two centuries and, in the 1-dimensional case, the large literature devoted to this topic has been summarized in several well known monographs (see for example [17], [18], [6], [9]). In this case, even if at analytical level many deep problems remain open, at the algebraic level the situation is well understood and described by Favard Lemma which, to any probability measure μ on the real line with finite moments of any order, associates two sequences, called the Jacobi sequences of μ ,

$$\{(\omega_n)_{n \in \mathbb{N}}, (\alpha_n)_{n \in \mathbb{N}}\}, \quad \omega_n \in \mathbb{R}_+, \alpha_n \in \mathbb{R}, \quad n = 0, 1, 2, \dots \quad (1.1)$$

subjected to the only constraint that, for any $n, k \in \mathbb{N}$,

$$\omega_n = 0 \implies \omega_{n+k} = 0, \quad (1.2)$$

and conversely, given two such sequences, it gives an inductive way to uniquely reconstruct:

- (i) a state on the algebra \mathcal{P} of polynomials in one indeterminate (see subsection 2.3),
- (ii) the orthogonal decomposition of \mathcal{P} canonically associated to this state (see section 3).

In this sense one can say that the pair of sequences (1.1), subjected to the only constraint (1.2), constitutes a full set of algebraic invariants for the equivalence classes of probability measures on the real line with respect to the equivalence relation $\mu \sim \nu$ if and only if all moments of μ and ν are finite and coincide (moment equivalence of probability measures on \mathbb{R}).

Compared to the 1-dimensional case the literature available in the multi-dimensional case is definitively scarce, even if several publications (see e.g [7], [10], [14], [15]) show an increasing interest to the problem in the past years, and for several years it has been mainly confined to applied journals, where it emerges in connection with different kinds of approximation problems. The need for an insightful theory was soon perceived by the mathematical community, for example in the 1953 monograph [8] (cited in [20]), the authors claim that " ... there does not seem to be an extensive general theory of orthogonal polynomials in several variables ... ".

Several progresses followed, both on the analytical front concerning multi-dimensional extensions of Carleman's criteria [16], [19], and on the algebraic front, with the introduction of the matrix approach [13] and the early formulations of the multi-dimensional Favard lemma [11], [12], [20].

However, even with these progresses in view, one cannot yet speak of a "general theory of orthogonal polynomials in several variables". In fact the currently adopted multi-dimensional formulations of Favard Lemma are based on two sequences of matrices, one of which rectangular, with quadratic constraints among the elements of these sequences. Such formulations look far from the elegant simplicity of the 1-dimensional Favard lemma. In fact the multi-dimensional analogues of positive, resp. real, numbers are the positive definite, resp. Hermitean, matrices. Therefore intuitively one would expect that a multi-dimensional extension of the Favard lemma would replace the (ω_n) -sequence by a sequence of positive definite matrices and the (α_n) -sequence by a sequence of Hermitean matrices. The precise formulation of this naive conjecture is what we call *the multi-dimensional Favard problem* (see section 3).

The goal of the present paper is to prove that the above mentioned naive generalization of Favard lemma is possible. This possibility was hinted, and heavily relies, on the quantum probabilistic approach to the theory of orthogonal polynomials first proposed, in the 1-dimensional case, in the paper [1], where the notion of quantum decomposition of a classical random variable was introduced and used to establish a canonical identification between the theory of orthogonal polynomials in 1 indeterminate and the theory of 1-mode interacting Fock spaces (IFS). One can say that the quantum decomposition of a classical random variable is a re-formulation of the Jacobi recurrence relation.

The early extensions of this approach to the multi-dimensional case [5], [2] constructed the quantum decomposition of the coordinate random variables in terms of creation, annihilation and preservation operators on an IFS canonically associated to the orthogonal decomposition of the polynomial algebra in d indeterminates \mathcal{P}_d with respect to a given state, however they still relied on the use of rectangular matrices.

An important step towards the solution of Favard problem for polynomials in d indeterminates ($d \in \mathbb{N}$) was done in the paper [4] where it was proved that the reconstruction of the state on \mathcal{P}_d can be achieved using only the commutators between creation and annihilation operators and the preservation operator. These operators preserve the orthogonal gradation, therefore each of them is determined by a sequence of square matrices. Moreover the preservation operator, being symmetric, is determined by a sequence

of Hermitean matrices while the commutators between creation and annihilation operators are determined by two positive definite matrix valued kernels (respectively $(a_j a_k^+)$ and $(a_k^+ a_j)$, $(j, k \in \{1, \dots, d\})$ whose restriction to the orthogonal gradation define two sequences of positive definite matrices.

Although this framework is much nearer to the one conjectured in the Favard problem, yet important discrepancies remain, in particular:

- (i) while the sequence of Hermitean matrices is only one for each coordinate random variable, as conjectured, the commutators involved are defined by two sequences of positive definite matrices;
- (ii) the dimensions of the positive definite matrices in item (i) are much higher than those of the corresponding Hermitean matrices;
- (iii) contrarily to the 1-dimensional case, the correspondence between IFS and families of orthogonal polynomials is not one-to-one;
- (iv) the multi-dimensional analogue of the compatibility condition (1.2) is involved and not easy to interpret.

The main results of the present paper are:

- (1) the identification of the theory of orthogonal polynomials functions on \mathbb{R}^d with the theory of **symmetric** interacting Fock spaces over \mathbb{C}^d (see section 8);
- (2) as a corollary of statement (1) above, the positive answer to the Favard problem;
- (3) the identification of the multi-dimensional extension of the compatibility condition (1.2) with the condition for the existence of the creator in an IFS.

The usual Boson Fock space corresponds to the case in which all the matrices in the principal Jacobi sequence (the positive definite ones) are identity matrices and all the Hermitean matrices in the secondary Jacobi sequence are zero. This corresponds to the quantum probabilistic characterization of the standard Gaussian measure in terms of commutators obtained in [4] and to the fact that the commutation relations, canonically associated to the orthogonal polynomial gradation induced by this measure, are the Heisenberg CCR for a system with finitely many degrees of freedom. In the present approach the emergence of the symmetric tensor algebra as well as of nontrivial commutation relations are both consequences of the commutativity of the coordinate random variables. In this sense a non commutative structure is canonically deduced from a commutative one.

The above results naturally suggest the program of a purely algebraic classification of the moment equivalence classes of probability measures on \mathbb{R}^d and provide the basic tools for its realization. From the point of view of physics the mathematical clarification of the structure of the usual Boson Fock space within the more general and traditional theory of orthogonal polynomials, with the important addition of the quantum decomposition and the consequent clarification of the probabilistic origins of the commutation relations, can open the way to the investigation of the possible nonlinear generalizations

of first and second quantization, a field which in quantum probability is already object of investigation since a few years. Also the program, initiated in [3] of a purely algebraic characterization of probability measures on \mathbb{R}^d with all moments, can receive a new impetus from the present results, but this topic will be discussed elsewhere.

2 The polynomial algebra in d commuting indeterminates

2.1 Notations

In the following we fix a finite set

$$D \equiv \{1, \dots, d\}, \quad d \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$$

and we denote

$$\mathcal{P} := \mathbb{C}[(X_j)_{j \in D}] \quad (2.1)$$

the complex polynomial algebra in the commuting indeterminates $(X_j)_{j \in D}$ with the $*$ -algebra structure uniquely determined by the prescription that the X_j are self-adjoint. In the present paper d will be fixed and in the following it will be frequently omitted from the notations, in particular we will simply write \mathcal{P} instead of \mathcal{P}_d or \mathcal{P}_D .

Unless otherwise specified, algebras and vector spaces will be complex.

For any vector space V we denote $\mathcal{L}(V)$ the algebra of linear maps of V into itself.

For $F = \{1, \dots, m\} \subset D$ and $v = (v_1, \dots, v_m) \in \mathbb{C}^m$ we will use the notation:

$$X_v := \sum_{j \in F} v_j X_j$$

The principle of identity of polynomial states that a polynomial is identically zero if and only if all its coefficients are zero. This is equivalent to say that the generators X_j ($j \in D$) are algebraically independent. These generators will also be called *coordinates*. A *monomial of degree $n \in \mathbb{N}$* is by definition any product of the form

$$M := \prod_{j \in F} X_j^{n_j} \quad (2.2)$$

where $F \subseteq D$ is a finite subset, and for any $j \in F$, $n_j \in \mathbb{N}$

$$\sum_{j \in F} n_j = n$$

A monomial of the form (2.2) is said to be *localized in the subset $F \subseteq D$* .

The algebra generated by such monomials will be denoted

$$\mathcal{P}_F \subseteq \mathcal{P} := \mathcal{P}_D$$

Notice that, with this definition of localization, if $F \subseteq G \subseteq D$ then any monomial localized in F is also localized in G , i.e.

$$\mathcal{P}_F \subseteq \mathcal{P}_G \subseteq \mathcal{P}$$

For all $n \in \mathbb{N}$ and for any subset $F \subseteq D$, we use the following notations:

$$\mathcal{M}_{F,n] := \{\text{the set of monomials of degree less or equal than } n \text{ localized in } F\} \quad (2.3)$$

$$\mathcal{M}_{F,n} := \{\text{the set of monomials of degree } n \text{ localized in } F\} \quad (2.4)$$

$$\mathcal{P}_{F,n]} := \{\text{the vector subspace of } \mathcal{P} \text{ generated by the set } \mathcal{M}_{F,n]}\} \quad (2.5)$$

$$\mathcal{P}_{F,n}^0 := \{\text{the vector subspace of } \mathcal{P} \text{ generated by the set } \mathcal{M}_{F,n}\} \quad (2.6)$$

with the convention that

$$X_j^0 = 1_{\mathcal{P}}, \quad \forall j \in D$$

where $1_{\mathcal{P}}$ denotes the identity of \mathcal{P} . We use the apex 0 in $\mathcal{P}_{F,n}^0$ to distinguish the monomial gradation (see (2.11) below), which is purely algebraic, from the orthogonal gradations, which will be introduced later on and depend on the choice of a state on \mathcal{P} . The only monomial of degree $n = 0$ is by definition

$$M_0 := 1_{\mathcal{P}}$$

Therefore

$$\mathcal{P}_{F,0}^0 = \mathcal{P}_{F,0]} = \mathbb{C} \cdot 1_{\mathcal{P}} \quad (2.7)$$

More generally, if $|F| = m$ then for any $n \in \mathbb{N}$ there are exactly

$$d_n := \binom{n+m-1}{m-1} \quad (2.8)$$

monomials of degree n localized in F and, by the principle of identity of polynomials they are linearly independent. Therefore one has

$$\mathcal{P}_{F,n}^0 \equiv \mathbb{C}^{d_n}$$

where the isomorphism is meant in the sense of vector spaces.

For future use it is useful to think of \mathcal{P} as an algebra of operators acting on itself by left multiplication. In the following, when no confusion is possible, we will use the same symbol for an element $Q \in \mathcal{P}$ and for its multiplicative action on \mathcal{P} . Sometimes, to emphasize the fact that Q is considered as an element of the vector space \mathcal{P} , we will use the notation

$$Q \cdot 1_{\mathcal{P}}$$

The sequence $(\mathcal{P}_{F,n])}_{n \in \mathbb{N}}$ is an increasing filtration of complex finite dimensional $*$ -vector subspaces of \mathcal{P} , i.e:

$$\mathcal{P}_{F,0]} \subset \mathcal{P}_{F,1]} \subset \mathcal{P}_{F,2]} \subset \cdots \subset \mathcal{P}_{F,n]} \subset \cdots \subset \mathcal{P}_F \subset \mathcal{P} \quad (2.9)$$

Moreover

$$\bigcup_{n \in \mathbb{N}} \mathcal{P}_{F,n]} = \mathcal{P}_F \quad (2.10)$$

and, for any $m, n \in \mathbb{N}$ one has

$$\mathcal{P}_{F,m]} \cdot \mathcal{P}_{F,n]} = \mathcal{P}_{F,m+n]}$$

The sequence $(\mathcal{P}_{F,n}^0)_{n \in \mathbb{N}}$ defines a vector space gradation of \mathcal{P}_F

$$\mathcal{P}_F = \sum_{k \in \mathbb{N}} \mathcal{P}_{F,k}^0 \quad (2.11)$$

called the monomial decomposition of \mathcal{P} . In (2.11) the symbol $\dot{\sum}$ denotes direct sum in the sense of vector spaces, i.e. elements of \mathcal{P} are finite linear sums of elements in some of the $\mathcal{P}_{F,n}^0$ and

$$m \neq n \implies \mathcal{P}_{F,m}^0 \cap \mathcal{P}_{F,n}^0 = \{0\} \quad (2.12)$$

The gradation (2.11) is compatible with the filtration $(\mathcal{P}_{F,n})$ in the sense that, for any $n \in \mathbb{N}$,

$$\mathcal{P}_{F,n]} = \dot{\sum}_{k \in \{0,1,\dots,n\}} \mathcal{P}_{F,k}^0 \quad (2.13)$$

Let $(e_j)_{j \in D}$ be a linear basis of \mathbb{C}^d . The coordinate X_j ($j \in D$) defines a linear map

$$X : v = \sum_{j \in D} v_j e_j \in \mathbb{C}^d \longmapsto X_v := \sum_{j \in D} v_j X_j \in \mathcal{L}(\mathcal{P})$$

Lemma 2.1 *Let $W \subset \mathcal{P}$ be a vector subspace. Then the set*

$$XW := \{X_v W : v \in \mathbb{C}^m\} \quad (2.14)$$

is a vector subspace of \mathcal{P} .

PROOF. The set (2.14) coincides with the set

$$\left\{ \sum_{j \in F} X_j \xi_w^{(j)} : \xi_w^{(j)} \in W, \forall j \in F \right\}$$

and this is clearly a vector space. ■

Lemma 2.2 *In the notation (2.14), for each $n \in \mathbb{N}$, one has*

$$X\mathcal{P}_{F,n}^0 = \mathcal{P}_{F,n+1}^0 \quad (2.15)$$

$$\mathcal{P}_{F,n+1]} = X\mathcal{P}_{F,n]} \dot{+} \mathcal{P}_{F,0]} = \mathcal{P}_{F,n]} \dot{+} \mathcal{P}_{F,n+1}^0 \quad (2.16)$$

where $\dot{+}$ denotes direct sum in the sense of vector spaces.

PROOF. Since $\mathcal{M}_{F,n}$ is a linear basis of $\mathcal{P}_{F,n}^0$, $\bigcup_{j \in F} X_j \mathcal{M}_{F,n} \subset \mathcal{P}_{F,n+1}^0$ is a system of generators of the subspace $X\mathcal{P}_{F,n}^0$. Hence $X\mathcal{P}_{F,n}^0 \subset \mathcal{P}_{F,n+1}^0$. The converse inclusion is clear because $\bigcup_{j \in F} X_j \mathcal{M}_{F,n}$ is also a system of generators of $\mathcal{P}_{F,n+1}^0$. This proves (2.15). (2.16) follows from (2.13) and (2.15). ■

Notations: In the following the set D will be fixed and we will use the notations:

$$\mathcal{P}_D = \mathcal{P}, \quad \mathcal{P}_n^0 := \mathcal{P}_{D,n}^0, \quad \mathcal{P}_{n]} := \mathcal{P}_{D,n]}, \quad n \in \mathbb{N}$$

with the convention

$$\mathcal{P}_{-1}^0 = \mathcal{P}_{-1]} = \{0\}$$

Lemma 2.3 For $n \in \mathbb{N}$, let \mathcal{P}_{n+1} be a vector subspace of $\mathcal{P}_{n+1}]$ such that

$$\mathcal{P}_{n]} \dot{+} \mathcal{P}_{n+1} = \mathcal{P}_{n+1}] \quad (2.17)$$

Then as a vector space \mathcal{P}_{n+1} is isomorphic to \mathcal{P}_{n+1}^0

PROOF. Since the sum in (2.17) is direct, one has

$$\dim(\mathcal{P}_{n+1}^0) = \dim(\mathcal{P}_{n+1}]) - \dim(\mathcal{P}_{n]}) = \dim(\mathcal{P}_{n+1})$$

■

The real linear span $\mathcal{P}_{\mathbb{R}}$ of the generators X_j induces a natural real structure on \mathcal{P} given by

$$\mathcal{P} = \mathcal{P}_{\mathbb{R}} + i\mathcal{P}_{\mathbb{R}} \quad (2.18)$$

where the sum in (2.18) is direct in the real vector space sense.

Remark. All the properties considered in this section continue to hold if one restricts one's attention to the real algebra $\mathcal{P}_{\mathbb{R}}$.

2.2 \mathcal{P} and the symmetric tensor algebra

In the following \otimes will denote algebraic tensor product and $\widehat{\otimes}$ its symmetrization. The tensor algebra over \mathbb{C}^d is the vector space

$$\mathcal{T}(\mathbb{C}^d) := \sum_{n \in \mathbb{N}} (\mathbb{C}^d)^{\otimes n}$$

with multiplication given by

$$(u_n \otimes \cdots \otimes u_1) \otimes (v_m \otimes \cdots \otimes v_1) := u_n \otimes \cdots \otimes u_1 \otimes v_m \otimes \cdots \otimes v_1$$

for any $m, n \in \mathbb{N}$ and all $u_j, v_j \in \mathbb{C}^d$. The extension to \mathbb{C}^d of the natural real structure on \mathbb{C} given by $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ and the associated involution, induces a $*$ -algebra structure on $\mathcal{T}(\mathbb{C}^d)$ whose involution is characterized by the property that

$$(v_n \otimes \cdots \otimes v_1)^* := v_n \otimes \cdots \otimes v_1, \quad \forall n \in \mathbb{N}, \forall v \in \mathbb{R}^d \quad (2.19)$$

The $*$ -sub-algebra of $\mathcal{T}(\mathbb{C}^d)$ generated by the elements of the form

$$v^{\otimes n} := v \otimes \cdots \otimes v \quad (n\text{-times}), \quad \forall n \in \mathbb{N}, \forall v \in \mathbb{C}^d$$

is called the symmetric tensor algebra over \mathbb{C}^d and denoted $\mathcal{T}_{sym}(\mathbb{C}^d)$.

Lemma 2.4 Let $(e_j)_{j \in D}$ be a linear basis of \mathbb{C}^d . Then, for all $n \in \mathbb{N}^*$, the map

$$e_{j_n} \widehat{\otimes} \cdots \widehat{\otimes} e_{j_1} \mapsto X_{j_n} \cdots X_{j_1} \quad (2.20)$$

where (j_1, \dots, j_n) varies among all the maps $j : \{1, \dots, n\} \rightarrow \{1, \dots, d\}$, extends to a vector space isomorphism

$$S_n^0 : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n^0$$

with

$$S_0^0 : z \in (\mathbb{C}^d)^{\widehat{\otimes} 0} \equiv \mathbb{C} \mapsto z1_{\mathcal{P}} \in \mathcal{P}_0^0 = \mathbb{C}1_{\mathcal{P}}$$

and the map

$$S^0 := \sum_{n \in \mathbb{N}} S_n^0 : \mathcal{T}_{sym}(\mathbb{C}^d) := \sum_{n \in \mathbb{N}} (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \sum_{n \in \mathbb{N}} \mathcal{P}_n^0 \equiv \mathcal{P}$$

is a gradation preserving $*$ -algebra isomorphism.

PROOF. The map (2.20) is well defined because both sides are invariant under permutations. It is a vector space isomorphism because it maps a linear basis onto a linear basis. The second statement is clear given the first one. \blacksquare

Lemma 2.5 *Let $(\mathcal{P}_n)_{n \in \mathbb{N}}$ be any family of subspaces of \mathcal{P} such that*

$$\mathcal{P}_{k+1] = \mathcal{P}_{k]} \dot{+} \mathcal{P}_{k+1}, \quad \forall k \in \mathbb{N}$$

$$\mathcal{P}_0 = \mathcal{P}_{0]} = \mathcal{P}_0^0 = \mathbb{C}1_{\mathcal{P}}$$

Then, for all $n \in \mathbb{N}$, there exists a vector space isomorphism

$$S_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n \tag{2.21}$$

and the map

$$S := \sum_{n \in \mathbb{N}} S_n : \mathcal{T}_{sym}(\mathbb{C}^d) := \sum_{n \in \mathbb{N}} (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \sum_{n \in \mathbb{N}} \mathcal{P}_n \equiv \mathcal{P}$$

is a gradation preserving $*$ -algebra isomorphism.

PROOF. From Lemma 2.3 we know that, for all $n \in \mathbb{N}$, \mathcal{P}_n has the same dimension as \mathcal{P}_n^0 which is given by (2.8). Hence there exists a vector space isomorphism

$$T_n : \mathcal{P}_n^0 \rightarrow \mathcal{P}_n, \quad \forall n \in \mathbb{N}.$$

Defining $S_n := T_n \circ S_n^0$ where S_n^0 is given by (2.20), the thesis follows. \blacksquare

2.3 States on \mathcal{P}

For the terminology on pre-Hilbert spaces we refer to Appendix 7. Denote $\mathcal{S}(\mathcal{P})$ the set of states on \mathcal{P} . Any state $\varphi \in \mathcal{S}(\mathcal{P})$ defines a pre-scalar product $\langle \cdot, \cdot \rangle_{\varphi}$ on \mathcal{P} given by

$$(a, b) \in \mathcal{P} \times \mathcal{P} \mapsto \langle a, b \rangle_{\varphi} := \varphi(a^*b) \in \mathbb{C} \tag{2.22}$$

satisfying the normalization condition

$$\langle 1_{\mathcal{P}}, 1_{\mathcal{P}} \rangle_{\varphi} = 1 \tag{2.23}$$

Since \mathcal{P} is a commutative $*$ -algebra with symmetric algebraic generators, such a pre-scalar product satisfies the additional conditions

$$\langle a, b \rangle_\varphi \in \mathbb{R}, \quad \forall a, b \in \mathcal{P}_\mathbb{R} \quad (2.24)$$

$$\langle ab, c \rangle_\varphi = \langle b, a^*c \rangle_\varphi, \quad \forall a, b, c \in \mathcal{P} \quad (2.25)$$

where $\mathcal{P}_\mathbb{R}$ is given by (2.18) and a^* denotes the adjoint of a in \mathcal{P} . The pair

$$(\mathcal{P}, \langle \cdot, \cdot \rangle_\varphi)$$

is a commutative pre-Hilbert algebra.

Lemma 2.6 *For a pre-scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{P} the following statements are equivalent:*

- (i) $\langle \cdot, \cdot \rangle$ satisfies (2.23) and, for each $j \in D$, multiplication by the coordinate X_j is a symmetric linear operator on \mathcal{P} with respect to $\langle \cdot, \cdot \rangle$,
- (ii) $\langle \cdot, \cdot \rangle$ satisfies (2.23), (2.24) and (2.25),
- (iii) The pre-scalar product $\langle \cdot, \cdot \rangle$ is induced by a state φ on \mathcal{P} in the sense of (2.22).

PROOF. (i) \Rightarrow (ii). If multiplication by each X_j is symmetric with respect to $\langle \cdot, \cdot \rangle$, then the same is true for every monomial. Hence (2.25) follows from the linearity of the maps $\langle Q, \cdot \rangle$ for each $Q \in \mathcal{P}$. Given two monomials M and M' , symmetry and commutativity imply that

$$\langle M, M' \rangle = \langle 1_\mathcal{P}, MM' \rangle = \langle 1_\mathcal{P}, M'M \rangle = \langle M', M \rangle$$

thus the scalar product is real on the monomials and this implies (2.24).

(ii) \Rightarrow (iii). If (2.25) holds, then the linear functional on \mathcal{P}

$$\varphi : Q \in \mathcal{P} \longmapsto \varphi(Q) := \langle 1_\mathcal{P}, Q \cdot 1_\mathcal{P} \rangle = \langle 1_\mathcal{P}, Q \rangle$$

is positive because, for any $Q \in \mathcal{P}$, one has

$$\varphi(Q^*Q) := \langle 1_\mathcal{P}, Q^*Q \cdot 1_\mathcal{P} \rangle = \langle Q, Q \rangle \geq 0.$$

Therefore, if also (2.23) holds, φ is a state on \mathcal{P} which induces the pre-scalar product $\langle \cdot, \cdot \rangle$ in the sense of (2.22) because

$$\langle a, b \rangle_\varphi \equiv \varphi(a^*b) = \langle 1_\mathcal{P}, a^*b \cdot 1_\mathcal{P} \rangle = \langle a, b \rangle.$$

(iii) \Rightarrow (i). If (iii) holds, then

$$\langle X_j a, b \rangle = \varphi((X_j a)^* b) = \varphi(a^* X_j^* b) = \varphi(a^* X_j b) = \langle a, X_j b \rangle.$$

The remaining properties are clear. ■

3 The multi-dimensional Favard problem

Let be given a subspace $\mathcal{P}_n \subset \mathcal{P}_{n]}$ satisfying

$$\mathcal{P}_{n]} = \mathcal{P}_{n-1]} \dot{+} \mathcal{P}_n \quad (3.1)$$

(with the convention $\mathcal{P}_{-1]} = \{0\}$). Notice that, since $\mathcal{P}_{n]}$ contains all monomials of degree n and $\mathcal{P}_{n-1]}$ does not contain any polynomial of degree n , (3.1) implies that every such monomial M_n can be written in the form

$$M_n = P_{n-1]} + P_n, \quad P_{n-1]} \in \mathcal{P}_{n-1]} , \quad P_n \in \mathcal{P}_n$$

and this is equivalent to say that

$$M_n - P_{n-1]} = P_n \in \mathcal{P}_n \quad (3.2)$$

It is clear that, when M_n runs among all monomials of degree n , the polynomials (3.2) form a linear basis of \mathcal{P}_n .

Thus, if \mathcal{P}_n satisfies (3.1), then it must contain a linear basis of the form (3.2).

Definition 3.1 *For $n \in \mathbb{N}$ let be given a decomposition of $\mathcal{P}_{n]}$ of the form (3.1). Any linear basis of $\mathcal{P}_{n]}$ of the form (3.2) will be called a monic basis, or a perturbation of the monomial basis of order n in the coordinates $(X_j)_{j \in D}$.*

Let φ be a state on \mathcal{P} and denote

$$\langle \cdot , \cdot \rangle := \langle \cdot , \cdot \rangle_\varphi$$

the corresponding pre-scalar product. When no ambiguity is possible, the elements ξ of \mathcal{P} (resp. $\mathcal{P}_{n]}$, \mathcal{P}_n^0) satisfying

$$\langle \xi, \xi \rangle = 0$$

will be simply called *zero norm vectors* without explicitly mentioning the pre-scalar product (or the associated state φ). By the Schwartz inequality the set of zero norm vectors in \mathcal{P} (resp. $\mathcal{P}_{n]}$, \mathcal{P}_n^0), denoted \mathcal{N}_φ (resp. $\mathcal{N}_{\varphi, n]}$, $\mathcal{N}_{\varphi, n}$) is a $*$ -subspace satisfying

$$\mathcal{P}\mathcal{N}_{\varphi, n} \subseteq \mathcal{P}\mathcal{N}_{\varphi, n]} \subseteq \mathcal{P}\mathcal{N}_\varphi \subseteq \mathcal{N}_\varphi \quad (3.3)$$

In particular \mathcal{N}_φ is a $*$ -ideal of \mathcal{P} . The monomial decomposition (2.11) is compatible with the filtration $(\mathcal{P}_{F, n])}$ in the sense of (2.13), therefore

$$\mathcal{P} = \mathcal{P}_{n]} \dot{+} \left(\sum_{k > n} \mathcal{P}_k^0 \right), \quad \forall n \in \mathbb{N}$$

For reasons that will be clear in the reconstruction theorem of section 5 we want to keep the discussion at a pure vector space, rather than Hilbert space level. In particular we don't want to quotient out the zero norm vectors. Therefore, rather than the usual Gram-Schmidt orthonormalization procedure, we use its pre-Hilbert space variant, described in Appendix 7.

Lemma 3.2 *Let φ be a state on \mathcal{P} and denote $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_\varphi$ the associated pre-scalar product. Then there exists a gradation*

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} (\mathcal{P}_{n,\varphi}, \langle \cdot, \cdot \rangle_{n,\varphi}) \quad (3.4)$$

called a φ -orthogonal polynomial decomposition of \mathcal{P} , with the following properties:

(i) *(3.4) is orthogonal for the unique pre-scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{P} defined by the conditions:*

$$\begin{aligned} \langle \cdot, \cdot \rangle|_{\mathcal{P}_{n,\varphi}} &= \langle \cdot, \cdot \rangle_{n,\varphi}, & \forall n \in \mathbb{N} \\ \mathcal{P}_{m,\varphi} &\perp \mathcal{P}_{n,\varphi}, & \forall m \neq n \end{aligned}$$

(ii) *(3.4) is compatible with the filtration $(\mathcal{P}_{n|})$ in the sense that*

$$\mathcal{P}_{k|} = \bigoplus_{h \in \{0,1,\dots,k\}} \mathcal{P}_{h,\varphi}, \quad \forall k \in \mathbb{N}, \quad (3.5)$$

(iii) *the pre-scalar product $\langle \cdot, \cdot \rangle$, defined in item (i) above, is induced by a state on \mathcal{P} , i.e. satisfies the conditions of Lemma 2.6.*

Conversely, let be given:

(j) *a vector space direct sum decomposition of \mathcal{P}*

$$\mathcal{P} = \sum_{n \in \mathbb{N}} \mathcal{P}_n \quad (3.6)$$

such that $\mathcal{P}_0 = \mathbb{C} \cdot 1_{\mathcal{P}}$, and for each $n \in \mathbb{N}$, \mathcal{P}_n has a monic basis of degree n ,

(jj) *for all $n \in \mathbb{N}$ a pre-scalar product $\langle \cdot, \cdot \rangle_n$ on \mathcal{P}_n with the property that $1_{\mathcal{P}}$ has norm 1 and the unique pre-scalar product $\langle \cdot, \cdot \rangle$ on \mathcal{P} defined by the conditions:*

$$\begin{aligned} \langle \cdot, \cdot \rangle|_{\mathcal{P}_n} &= \langle \cdot, \cdot \rangle_n, & \forall n \in \mathbb{N} \\ \mathcal{P}_n &\perp \mathcal{P}_m, & \forall m \neq n \end{aligned}$$

satisfies the normalization condition (2.23) and multiplication by the coordinates X_j ($j \in D$) are $\langle \cdot, \cdot \rangle$ -symmetric linear operators on \mathcal{P} .

Then there exists a state φ on \mathcal{P} such that the decomposition (3.6) is the orthogonal polynomial decomposition of \mathcal{P} with respect to φ .

PROOF. In the above notations, for each $k \in \mathbb{N}$ define inductively the subspace $\mathcal{P}_{k,\varphi}$ and the two sequences of $\langle \cdot, \cdot \rangle$ -orthogonal projectors

$$P_{k|,\varphi} : \mathcal{P} \rightarrow \mathcal{P}_{k|}, \quad P_{k,\varphi} : \mathcal{P} \rightarrow \mathcal{P}_{k,\varphi}, \quad \forall k \in \mathbb{N}$$

as follows. For $k = 0$, define $\mathcal{P}_{0,\varphi} := \mathcal{P}_{0|}$ and

$$P_{0,\varphi} := P_{0|,\varphi} : Q \in \mathcal{P} \mapsto \varphi(Q)1_{\mathcal{P}} = \langle 1_{\mathcal{P}}, Q \cdot 1_{\mathcal{P}} \rangle 1_{\mathcal{P}} \in \mathcal{P}_{0|} =: \mathcal{P}_{0,\varphi}, \quad \forall Q \in \mathcal{P}$$

Having defined

$$\{P_{0,\varphi}, P_{1,\varphi}, \dots, P_{n,\varphi}\}, \quad \{P_{0],\varphi}, P_{1],\varphi}, \dots, P_{n],\varphi}\}$$

so that for each $k \in \{0, 1, \dots, n\}$ the space $\mathcal{P}_{k,\varphi}$ has a monic basis of order k and (3.5) is satisfied, in the notation (2.4), define

$$\mathcal{P}_{n+1,\varphi} := \text{lin-span}\{M_{n+1} - P_{n],\varphi}(M_{n+1}); M_{n+1} \in \mathcal{M}_{D,n+1}\} \quad (3.7)$$

Then by construction

$$\mathcal{P}_{n+1,\varphi} + \mathcal{P}_{n]} = \mathcal{P}_{n+1]}$$

and the space $\mathcal{P}_{n+1,\varphi}$ has a monic basis of order $n+1$, in particular

$$\mathcal{P}_{n+1,\varphi} \cap \mathcal{P}_{n]} = \{0\}$$

Applying Corollary 7.3 of Appendix 7 with $\mathcal{K} = \mathcal{P}$, $\mathcal{K}_0 = \mathcal{P}_{n+1,\varphi}$, \mathcal{K}_1 an arbitrary subspace of \mathcal{P} such that $\mathcal{P} = \mathcal{P}_{n+1,\varphi} \dot{+} \mathcal{K}_1$ and $\mathcal{K}_{0,1}$ any vector space supplement of the $\langle \cdot, \cdot \rangle$ -zero norm subspace $\mathcal{K}_{0,0}$ of $\mathcal{P}_{n+1,\varphi}$, we define the orthogonal projection

$$P_{n+1,\varphi} : \mathcal{P} \rightarrow \mathcal{P}_{n+1,\varphi}$$

which by construction is onto $\mathcal{P}_{n+1,\varphi}$ hence orthogonal to $\mathcal{P}_{n],\varphi}$. Therefore the operator

$$P_{n+1],\varphi} := P_{n],\varphi} + P_{n+1,\varphi}$$

is the orthogonal projection onto $\mathcal{P}_{n+1]}$. Finally, given φ , the conditions of Lemma 2.6 are satisfied by the associated pre-scalar product on \mathcal{P} . This completes the induction construction.

To prove the converse, notice that the fact that each \mathcal{P}_n has a monic basis of order n implies that the decomposition (3.6) satisfies condition (3.5). In fact this is true for \mathcal{P}_0 by construction and, supposing it true for $n \in \mathbb{N}$, the fact that \mathcal{P}_n has a monic basis of order n implies that \mathcal{P}_{n+1} contains all monomials of degree $n+1$ modulo an additive polynomial of degree $\leq n$. Thus the sum $\mathcal{P}_{n]} \dot{+} \mathcal{P}_{n+1]}$ contains all monomials of degree $\leq n+1$ hence, being a vector space, it coincides with $\mathcal{P}_{n+1]}$. Thus, by induction, property (3.5) holds for each $n \in \mathbb{N}$. Because of Lemma 2.6, condition (jj) implies that the pre-scalar product $\langle \cdot, \cdot \rangle$ is induced by a state φ in the sense of the identity (2.22). This implies that the decomposition (3.6) is the orthogonal polynomial decomposition of \mathcal{P} with respect to the state φ . \blacksquare

The following Lemma shows that the isomorphism, defined abstractly in Lemma 2.5 can be explicitly constructed if the gradation on \mathcal{P} is the one constructed in Lemma 3.2.

Lemma 3.3 *Let be given a vector space direct sum decomposition of \mathcal{P} of the form (3.6) satisfying conditions (j) and (jj) of Lemma 3.2. Let $B_n \subset \mathcal{P}_n$ be a perturbation of the monomial basis (see Definition 3.1) and for each monomial $M_n \in \mathcal{M}_{D,n}$ denote $p_n(M_n)$ the corresponding element of B_n . Then the map*

$$\pi_n : e_{j_n} \widehat{\otimes} e_{j_{n-1}} \widehat{\otimes} \dots \widehat{\otimes} e_{j_1} \in (\mathbb{C}^d)^{\widehat{\otimes} n} \longmapsto p_n(X_{j_n} X_{j_{n-1}} \dots X_{j_1}) \cdot 1_{\mathcal{P}} \in \mathcal{P}_n \quad (3.8)$$

where $n \in \mathbb{N}^*$ ($\pi_0 = id_{\mathbb{C}}$) and $\widehat{\otimes}$ denotes symmetric tensor product, extends to a vector space isomorphism.

PROOF. Let B_n be a monic basis of \mathcal{P}_n (which exists by assumption). Denoting $j : \{1, \dots, n\} \rightarrow \{1, \dots, d\}$ a generic function, the map

$$e_{j_n} \otimes e_{j_{n-1}} \otimes \dots \otimes e_{j_1} \longmapsto p_n(X_{j_n} X_{j_{n-1}} \dots X_{j_1}) \cdot 1_{\mathcal{P}} \in \mathcal{P}_n \quad (3.9)$$

is well defined on a linear basis of $(\mathbb{C}^d)^{\otimes n}$ because $X_{j_n} X_{j_{n-1}} \dots X_{j_1}$ is a monomial of degree n . Since both sides in (3.9) are multi-linear, by the universal property of the tensor product it extends to a linear map, denoted $\widehat{\pi}_n$, of $(\mathbb{C}^d)^{\otimes n}$ into \mathcal{P}_n . This map is surjective because when j runs over all maps $\{1, \dots, n\} \rightarrow \{1, \dots, d\}$, $p_n(X_{j_n} X_{j_{n-1}} \dots X_{j_1}) \cdot 1_{\mathcal{P}}$ runs over a linear basis of \mathcal{P}_n . Since the right hand side of (3.9) is invariant under permutations of the indices j_n, j_{n-1}, \dots, j_1 , $\widehat{\pi}_n$ induces a linear map of the vector space of equivalence classes of elements of $(\mathbb{C}^d)^{\otimes n}$ with respect to the equivalence relation induced by the linear action of the permutation group. Since this quotient space is canonically isomorphic to the symmetric tensor product $(\mathbb{C}^d)^{\widehat{\otimes} n}$, this induced map defines a linear extension of the map (3.8).

This extension is an isomorphism because we have already proved that surjectivity and injectivity follow from the fact that the equivalence class under permutations of any n -tuple $(j_n, j_{n-1}, \dots, j_1)$ defines a unique element of the basis $\{p_n(M_n) \cdot 1_{\mathcal{P}}; M_n \in \mathcal{P}_n\}$ of \mathcal{P}_n . \blacksquare

Remark. The construction of Lemma 3.2 depends on the choice of the vector space supplement of the zero norm subspace of $\mathcal{P}_{n,\varphi}$. However any vector in another supplement will differ by a zero norm vector from a vector in the previous choice. Therefore, at Hilbert space level, the two choices will coincide.

3.1 Statement of the multi-dimensional Favard problem

From Lemma 3.2 we know that the orthogonal polynomial decomposition of \mathcal{P} with respect to a state φ induces a decomposition of \mathcal{P} of the form (3.6). Given such a decomposition, for every $n \in \mathbb{N}$, we can use the vector space isomorphisms π_n defined in Lemma 3.3 to transfer the pre-Hilbert structure of \mathcal{P}_n on the symmetric tensor product space $(\mathbb{C}^d)^{\widehat{\otimes} n}$. Imposing the orthogonality of the \mathcal{P}_n 's one obtains a gradation preserving unitary isomorphism between \mathcal{P} , with the orthogonal polynomial gradation induced by the state φ , and a symmetric interacting Fock space structure over \mathbb{C}^d (see Appendix 8). The converse of this statement constitutes the essence of what we call the multi-dimensional Favard problem, namely:

Given a symmetric interacting Fock space structure over \mathbb{C}^d (see Lemma 8.5):

$$\bigoplus_{n \in \mathbb{N}} \left((\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \Omega_n \cdot \rangle_{\widehat{\otimes} n} \right)$$

- (i) does there exist a state φ on \mathcal{P} whose associated symmetric IFS is the given one?
- (ii) it is possible to parameterize all solutions of problem (i) and to characterize them constructively?

The second part of the present paper is devoted to the proof of the fact that both the above stated problems have a positive solution. Before that, in the following section, we establish some notations and necessary conditions.

4 The symmetric Jacobi relations

4.1 The three term recurrence relations

In this section we fix a state φ on \mathcal{P} and we follow the notations of Lemma 3.2 with the exception that we omit the index φ . Thus we write $\langle \cdot, \cdot \rangle$ for the pre-scalar product $\langle \cdot, \cdot \rangle_\varphi$, $P_{k]} : \mathcal{P} \rightarrow \mathcal{P}_{k]}$ ($k \in \mathbb{N}$) for the $\langle \cdot, \cdot \rangle$ -orthogonal projector in the pre-Hilbert space sense (see the proof of Lemma 3.2), \mathcal{P}_{k+1} for the space defined by (3.7) and

$$P_n = P_{n]} - P_{n-1]} \quad (4.1)$$

the corresponding projector. We know that

$$P_{n]}(\mathcal{P}_{\mathbb{R}}) \subseteq \mathcal{P}_{\mathbb{R}} \cap \mathcal{P}_{n]} = \mathcal{P}_{\mathbb{R}, n]}, \quad \forall n \in \mathbb{N} \quad (4.2)$$

and that the sequence $(\mathcal{P}_{n])}_{n \in \mathbb{N}}$ is an increasing filtration with union \mathcal{P} (see (2.9) and (2.10)). It follows that the sequence of projections (4.1) is a partition of the identity in $(\mathcal{P}, \langle \cdot, \cdot \rangle)$, i.e.

$$P_n P_m = \delta_{mn} P_m, \quad P_n = P_n^*, \quad \forall m, n \in \mathbb{N} \quad (4.3)$$

$$\sum_n P_n = \lim_n P_{n]} = 1_{\mathcal{P}}. \quad (4.4)$$

Lemma 4.1 *Suppose that, for some $m \in \mathbb{N}$, $P_m = 0$. Then*

$$P_n = 0, \quad \forall n \geq m. \quad (4.5)$$

PROOF. From (2.16) and the definition of P_{n+1} one deduces that

$$P_{n+1} = 0 \iff \mathcal{P}_{n+1]} = \mathcal{P}_{0]} \dot{+} \sum_{j \in D} X_j \mathcal{P}_{n]} = \mathcal{P}_{n]}.$$

Iterating the identity

$$X_j \mathcal{P}_{n]} \subseteq \mathcal{P}_{n]}, \quad \forall j \in D,$$

we see that, for any $k \geq 0$ and for any function $j : \{1, \dots, k\} \rightarrow D$, one has

$$X_{j_1} \cdots X_{j_k} \mathcal{P}_{n]} \subseteq \mathcal{P}_{n]}$$

and this implies that, for any $k \geq 0$,

$$\mathcal{P}_{n+k]} \subseteq \mathcal{P}_{n]}.$$

Therefore, since $(\mathcal{P}_{n])}_{n \in \mathbb{N}}$ is an increasing filtration, for any $k \geq 0$,

$$\mathcal{P}_{n+k]} = \mathcal{P}_{n]}$$

and this is equivalent to (4.5). ■

Theorem 4.2 *With the notation*

$$P_{-1]} := 0,$$

for any $j \in D$ and any $n \in \mathbb{N}$, one has

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n. \quad (4.6)$$

PROOF. Because of (4.4), for any $j \in D$,

$$X_j = 1_{\mathcal{P}} \cdot X_j \cdot 1_{\mathcal{P}} = \sum_{m,n \in \mathbb{N}} P_m X_j P_n.$$

Therefore

$$X_j P_n = \sum_{m \in \mathbb{N}} P_m X_j P_n.$$

The basic remark is that

$$X_j \mathcal{P}_n \subseteq \mathcal{P}_{n+1]},$$

i.e.

$$X_j P_n = P_{n+1]} X_j P_n.$$

Thus, if $m > n + 1$ (or equivalently $m - 1 \geq n + 1$), then

$$P_m] P_{n+1]} = P_{m-1]} P_{n+1]} = P_{n+1]}.$$

Hence

$$P_m X_j P_n = P_m P_{n+1]} X_j P_n = (P_m] - P_{m-1]}) P_{n+1]} X_j P_n = 0.$$

This proves that

$$m > n + 1 \implies P_m X_j P_n = 0.$$

Therefore, if $m < n - 1$, then

$$P_m X_j P_n = (P_n X_j P_m)^* = 0.$$

Summing up: $P_m X_j P_n$ can be non-zero only if $m \in \{n - 1, n, n + 1\}$ and this proves (4.6). ■

Definition 4.3 *The identity (4.6) is called the symmetric Jacobi relation.*

4.2 The CAP operators and the quantum decomposition

For each $n \in \mathbb{N}$ and $j \in D$, define the operators

$$\begin{aligned} a_{j|n}^+ &:= P_{n+1} X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \mathcal{P}_{n+1} \\ a_{j|n}^0 &:= P_n X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \mathcal{P}_n \\ a_{j|n}^- &:= P_{n-1} X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \longrightarrow \mathcal{P}_{n-1} \end{aligned} \quad (4.7)$$

Notation: if $v = (v_1, \dots, v_d) \in \mathbb{C}^d$, we denote

$$a_{v|n}^\varepsilon := \sum_{j \in D} v_j a_{j|n}^\varepsilon, \quad \varepsilon \in \{+, 0, -\}. \quad (4.8)$$

Notice that, D being a finite set, the spaces \mathcal{P}_n are finite dimensional. Moreover, in the present algebraic context, the sum

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n \quad (4.9)$$

is orthogonal and meant in the weak sense, i.e. for each element $Q \in \mathcal{P}$ there is a finite set $I \subset \mathbb{N}$ such that

$$Q = \sum_{n \in I} p_n, \quad p_n \in \mathcal{P}_n. \quad (4.10)$$

Theorem 4.4 *On \mathcal{P} , for any $j \in D$, the following operators are well defined*

$$\begin{aligned} a_j^+ &:= \sum_{n \in \mathbb{N}} a_{j|n}^+ \\ a_j^0 &:= \sum_{n \in \mathbb{N}} a_{j|n}^0 \\ a_j^- &:= \sum_{n \in \mathbb{N}} a_{j|n}^- \end{aligned}$$

and one has

$$X_j = a_j^+ + a_j^0 + a_j^- \quad (4.11)$$

in the sense that both sides of (4.11) are well defined on \mathcal{P} and the equality holds.

PROOF. For all $j \in D$ and all $Q \in \mathcal{P}$ (which is in the form (4.10)), one has

$$\begin{aligned} (a_j^+ + a_j^0 + a_j^-)Q &= \sum_{n \in \mathbb{N}} (a_{j|n}^+ + a_{j|n}^0 + a_{j|n}^-)Q \\ &= \sum_{n \in \mathbb{N}} (P_{n+1}X_jP_n + P_nX_jP_n + P_{n-1}X_jP_n)Q \\ &= \sum_{n \in \mathbb{N}} (X_jP_n)Q = X_j \sum_{n \in \mathbb{N}} P_nQ = X_jQ. \end{aligned}$$

According to the observation before Theorem 4.4, we notice that all the sums are finite. ■

4.3 Properties of the quantum decomposition

Notice that, by construction, for any $j \in D$ and $n \in \mathbb{N}$, the maps

$$a_{j|n}^+ := P_{n+1}X_jP_n$$

satisfy

$$a_{j|n}^+(\mathcal{P}_n) \subseteq \mathcal{P}_{n+1} \quad (4.12)$$

and recall that, by construction, the non-zero elements of \mathcal{P}_{n+1} are polynomials of degree $n+1$.

Lemma 4.5 For any $n \in \mathbb{N}$, denote

$$\mathcal{P}_{0,n+1} := \text{lin-span } \{a_{j|n}^+(\mathcal{P}_n); j \in D\}.$$

Then $\mathcal{P}_{0,n+1} = \mathcal{P}_{n+1}$.

PROOF. From (4.12) it follows that

$$\mathcal{P}_{0,n+1} \subseteq \mathcal{P}_{n+1}.$$

Suppose, by contradiction, that the inclusion is proper for some $n \in \mathbb{N}$. Then, there exists $\xi_{n+1} \in \mathcal{P}_{n+1} \setminus \{0\}$ such that

$$\xi_{n+1} \perp a_{j|n}^+(\mathcal{P}_n), \quad \forall j \in D.$$

Being in \mathcal{P}_{n+1} , ξ_{n+1} is orthogonal to $\mathcal{P}_{n]} = \bigoplus_{k=0}^n \mathcal{P}_k$. Therefore, for any $\xi_n \in \mathcal{P}_n$,

$$\xi_{n+1} \perp (a_{j|n}^+ \xi_n + a_{j|n}^0 \xi_n + a_{j|n}^- \xi_n), \quad \forall j \in D.$$

Then, due to the quantum decomposition (4.11), this is equivalent to

$$\xi_{n+1} \perp X_j \mathcal{P}_n, \quad \forall j \in D.$$

Equivalently, for any $\xi \in \mathcal{P}$, ξ_{n+1} is orthogonal to

$$X_j P_n \xi = X_j (P_n] - P_{n-1]) \xi = X_j P_n] \xi - X_j P_{n-1}] \xi, \quad \forall j \in D.$$

But, $X_j P_{n-1}] \xi \in \mathcal{P}_{n]}$, which is orthogonal to ξ_{n+1} . Therefore

$$\xi_{n+1} \perp X_j P_n] \xi, \quad \forall j \in D, \forall \xi \in \mathcal{P}$$

and, since surely $\xi_{n+1} \perp \mathcal{P}_0$, this and Lemma 2.2 yield

$$\xi_{n+1} \perp \sum_{j \in D} X_j \mathcal{P}_{n]} \dot{+} \mathcal{P}_{0]} = \mathcal{P}_{n+1]}.$$

Since $\xi_{n+1} \in \mathcal{P}_{n+1} \subseteq \mathcal{P}_{n+1]}$, this is possible if and only if $\xi_{n+1} = 0$, against the assumption. ■

Lemma 4.6 For any $j \in D$ and $n \in \mathbb{N}$, one has

$$\begin{aligned} (a_{j|n}^+)^* &= a_{j|n+1}^-, & (a_j^+)^* &= a_j^-, \\ (a_{j|n}^0)^* &= a_{j|n}^0, & (a_j^0)^* &= a_j^0. \end{aligned}$$

PROOF. For an arbitrary $j \in D$ and $n \in \mathbb{N}$ we have

$$(a_{j|n}^+)^* = (P_{n+1} X_j P_n)^* = P_n X_j P_{n+1} = a_{j|n+1}^-.$$

Recall that, with the notation (4.7),

$$a_{j|n}^- = P_{n-1} X_j P_n : \mathcal{P}_n \longrightarrow \mathcal{P}_{n-1}.$$

Thus

$$(a_j^+)^* = \left(\sum_{n \in \mathbb{N}} a_{j|n}^+ \right)^* = \sum_{n \in \mathbb{N}} (a_{j|n}^+)^* = \sum_{n \in \mathbb{N}} a_{j|n+1}^-$$

and, with the change of variables $n + 1 =: m \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, this becomes

$$(a_j^+)^* = \sum_{m \in \mathbb{N}^*} a_{j|m}^- = \sum_{n \in \mathbb{N}} a_{j|n}^- = a_j^-$$

because

$$a_{j|0}^- = 0.$$

Summing up

$$\begin{aligned} (a_j^+)^* &= a_j^-, & (a_j^-)^* &= ((a_j^+)^*)^* = a_j^+, \\ (a_{j|n}^0)^* &= (P_n X_j P_n)^* = a_{j|n}^0, \\ (a_j^0)^* &= \left(\sum_{n \in \mathbb{N}} a_{j|n}^0 \right)^* = \sum_{n \in \mathbb{N}} (a_{j|n}^0)^* = \sum_{n \in \mathbb{N}} a_{j|n}^0 = a_j^0. \end{aligned}$$

■

Lemma 4.7 *For any $j \in D$, the operators*

$$X_j, \quad a_j^+, \quad a_j^0$$

preserve the ideal \mathcal{N}_φ of zero norm vectors.

PROOF. It is sufficient to show that, for each $n \in \mathbb{N}$ if $\xi \in \mathcal{P}_n$ is a zero norm vector, then the same is true for the vectors

$$X_j \xi, \quad a_{j|n}^+ \xi, \quad a_{j|n}^0 \xi, \quad a_{j|n}^- \xi, \quad j \in D.$$

That $X_j \xi$ is a zero norm vector follows from

$$|\langle X_j \xi, X_j \xi \rangle| = |\langle X_j^2 \xi, \xi \rangle| \leq |\langle X_j^2 \xi, X_j^2 \xi \rangle|^{1/2} |\langle \xi, \xi \rangle|^{1/2} = 0.$$

From this and the quantum decomposition (4.11) it follows that the vector

$$X_j P_n \xi = a_{j|n}^+ \xi + a_{j|n}^0 \xi + a_{j|n}^- \xi$$

has zero norm. Since the right hand side is a sum of three mutually orthogonal vectors, it follows that each of them is a zero norm vector. ■

4.4 Commutation relations

In this section we briefly recall some known facts about commutation relations canonically associated to orthogonal polynomials (see [2]) which will be used in the following section. Given the quantum decompositions of the X_j 's:

$$X_j = a_j^+ + a_j^0 + a_j^-, \quad j \in \{1, \dots, d\},$$

one has, for each $j, k \in \{1, \dots, d\}$:

$$\begin{aligned}
0 &= [X_j, X_k] \\
&= [(a_j^+ + a_j^0 + a_j^-), (a_k^+ + a_k^0 + a_k^-)] \\
&= [a_j^+, a_k^+] + [a_j^+, a_k^0] + [a_j^+, a_k^-] + [a_j^0, a_k^+] \\
&\quad + [a_j^0, a_k^0] + [a_j^0, a_k^-] + [a_j^-, a_k^+] + [a_j^-, a_k^0] + [a_j^-, a_k^-].
\end{aligned} \tag{4.13}$$

This and the mutual orthogonality of the \mathcal{P}_k 's imply that, for each $j, k \in \{1, \dots, d\}$,

$$[a_j^+, a_k^+] = 0. \tag{4.14}$$

Taking the adjoint of which one obtains

$$[a_j^-, a_k^-] = 0.$$

(4.13) also implies that

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0. \tag{4.15}$$

Taking the adjoint of which one obtains

$$[a_j^0, a_k^-] + [a_j^-, a_k^0] = 0$$

Given the previous relations, (4.13) becomes equivalent to

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0. \tag{4.16}$$

(4.16) is equivalent to

$$[a_j^+, a_k^-] - [a_j^+, a_k^-]^* = -[a_j^0, a_k^0]. \tag{4.17}$$

In the following we will use only the mutual commutativity of the creators, i.e:

$$a_j^+ a_k^+ = a_k^+ a_j^+. \tag{4.18}$$

We refer the reader to [2] for more detailed analysis.

5 The reconstruction theorem

5.1 3-diagonal decompositions of \mathcal{P}

For two pre-Hilbert spaces \mathcal{H} and \mathcal{K} , we denote $\mathcal{L}_a(\mathcal{H}, \mathcal{K})$ the space of all adjointable linear operators from \mathcal{H} to \mathcal{K} (see Appendix 7).

Definition 5.1 For $n \in \mathbb{N}$, a 3-diagonal decomposition of $\mathcal{P}_{n]}$ is defined by:

(i) a vector space direct sum decomposition of $\mathcal{P}_{n]}$ such that

$$\mathcal{P}_{k]} = \sum_{h \in \{0, \dots, k\}} \mathcal{P}_h, \quad \forall k \in \{0, 1, \dots, n\}, \tag{5.1}$$

where each \mathcal{P}_k has a monic basis of order k ;

(ii) for each $k \in \{0, 1, \dots, n\}$, a pre-scalar product $\langle \cdot, \cdot \rangle_k$ on \mathcal{P}_k ,

(iii) two families of linear maps

$$v \in \mathbb{C}^d \mapsto a_{v|k}^+ \in \mathcal{L}_a(\mathcal{P}_k, \mathcal{P}_{k+1}), \quad k \in \{0, \dots, n-1\}, \quad (5.2)$$

$$v \in \mathbb{C}^d \mapsto a_{v|k}^0 \in \mathcal{L}_a(\mathcal{P}_k, \mathcal{P}_k), \quad k \in \{0, 1, \dots, n\}, \quad (5.3)$$

such that:

- for all $v \in \mathbb{R}^d$, $a_{v|k}^+$ maps the $(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k)$ -zero norm subspace into the $(\mathcal{P}_{k+1}, \langle \cdot, \cdot \rangle_{k+1})$ -zero norm subspace;
- for all $v \in \mathbb{R}^d$, $a_{v|k}^0$ is a self-adjoint operator on the pre-Hilbert space $(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k)$, thus in particular it maps the $(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k)$ -zero norm subspace into itself;
- denoting $*$ (when no confusion is possible) the adjoint of a linear map from $(\mathcal{P}_{k-1}, \langle \cdot, \cdot \rangle_{k-1})$ to $(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k)$ for any $k \in \{0, 1, \dots, n\}$, and defining $a_{v|-1}^+ = 0$ and

$$a_{v|k}^- := (a_{v|k-1}^+)^*, \quad k \in \{0, 1, \dots, n-1\}, \quad v \in \mathbb{C}^d, \quad (5.4)$$

the following identity is satisfied:

$$X_v \Big|_{\mathcal{P}_k} = a_{v|k}^+ + a_{v|k}^0 + a_{v|k}^-, \quad k \in \{0, 1, \dots, n-1\}, \quad v \in \mathbb{C}^d. \quad (5.5)$$

Remark.

- 1) In the following, if no confusion can arise, we will simply say that

$$\left\{ \left(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k \right)_{k=0}^n, \left(a_{\cdot|k}^+ \right)_{k=0}^{n-1}, \left(a_{\cdot|k}^0 \right)_{k=0}^n \right\} \quad (5.6)$$

is a 3-diagonal decomposition of $\mathcal{P}_{n]}$.

- 2) Condition (ii) above and the fact that the sum (5.1) is direct, imply that there exists a unique scalar product $\langle \cdot, \cdot \rangle_{n]}$ on $\mathcal{P}_{n]}$ such that

$$\langle \cdot, \cdot \rangle_{n]} \Big|_{\mathcal{P}_k} = \langle \cdot, \cdot \rangle_k, \quad \forall k \in \{0, 1, \dots, n\} \quad (5.7)$$

and the vector space decompositions (5.1) are orthogonal for $\langle \cdot, \cdot \rangle_{n]}$:

$$\mathcal{P}_{k]} = \bigoplus_{h \in \{0, \dots, k\}} \mathcal{P}_h, \quad \forall k \in \{0, 1, \dots, n\}. \quad (5.8)$$

Note that a priori all the objects defining a 3-diagonal decomposition of $\mathcal{P}_{n]}$ may depend on $n \in \mathbb{N}$.

Definition 5.2 (i) A 3-diagonal decomposition of $\mathcal{P}_{n+1}]$

$$\left\{ \left(\mathcal{P}_k(n+1), \langle \cdot, \cdot \rangle_{n+1,k} \right)_{k=0}^{n+1}, \left(a^+_{\cdot|k}(n+1) \right)_{k=0}^n, \left(a^0_{\cdot|k}(n+1) \right)_{k=0}^{n+1} \right\}$$

is called an extension of a 3-diagonal decomposition of $\mathcal{P}_n]$

$$\left\{ \left(\mathcal{P}_k(n), \langle \cdot, \cdot \rangle_{n,k} \right)_{k=0}^n, \left(a^+_{\cdot|k}(n) \right)_{k=0}^{n-1}, \left(a^0_{\cdot|k}(n) \right)_{k=0}^n \right\}$$

if

$$\begin{aligned} \mathcal{P}_k(n) &= \mathcal{P}_k(n+1), & \forall k \in \{0, \dots, n\} \\ \langle \cdot, \cdot \rangle_{n+1}] \Big|_{\mathcal{P}_n]} &= \langle \cdot, \cdot \rangle_n \\ a^0_{\cdot|k}(n+1) &= a^0_{\cdot|k}(n), & \forall k \in \{0, \dots, n\} \\ a^+_{\cdot|k}(n+1) &= a^+_{\cdot|k}(n), & \forall k \in \{0, \dots, n-1\}. \end{aligned}$$

(ii) A 3-diagonal decomposition of \mathcal{P} is a sequence of 3-diagonal decompositions

$$D_n := \left\{ \left(\mathcal{P}_k(n), \langle \cdot, \cdot \rangle_{n,k} \right)_{k=0}^n, \left(a^+_{\cdot|k}(n) \right)_{k=0}^{n-1}, \left(a^0_{\cdot|k}(n) \right)_{k=0}^n \right\}, \quad n \in \mathbb{N} \quad (5.9)$$

such that, for each $n \in \mathbb{N}$, D_{n+1} is an extension of D_n . In this case one simply writes

$$\left\{ \left(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k \right)_{k=0}^n, \left(a^+_{\cdot|k} \right)_{k=0}^{n-1}, \left(a^0_{\cdot|k} \right)_{k=0}^n \right\}_{n \in \mathbb{N}}. \quad (5.10)$$

Remark. Any 3-diagonal decomposition of $\mathcal{P}_n]$ induces, by restriction, a 3-diagonal decomposition of $\mathcal{P}_k]$ for any $k \leq n$.

In this section we discuss the following problem:
given a 3-diagonal decomposition of $\mathcal{P}_n]$, classify all its possible extensions.

Lemma 5.3 In the notations of Definition 5.1, for $k \in \mathbb{N}$, let be given:

(i) two vector subspaces $\mathcal{P}_{k-1} \subset \mathcal{P}_{k-1}]$, $\mathcal{P}_k \subset \mathcal{P}_k]$ with monic bases of order $k-1$ and k respectively and such that

$$\mathcal{P}_k] = \mathcal{P}_{k-1}] \dot{+} \mathcal{P}_k, \quad (5.11)$$

(ii) two arbitrary linear maps

$$v \in \mathbb{C}^d \mapsto A^0_{v|k} \in \mathcal{L}_a(\mathcal{P}_k, \mathcal{P}_k), \quad (5.12)$$

$$v \in \mathbb{C}^d \mapsto A^-_{v|k} \in \mathcal{L}_a(\mathcal{P}_k, \mathcal{P}_{k-1}). \quad (5.13)$$

Then, defining for any $v \in \mathbb{C}^d$ the map

$$A^+_{v|k} := X_v \Big|_{\mathcal{P}_k} - A^0_{v|k} - A^-_{v|k}, \quad (5.14)$$

the set

$$\mathcal{P}_{k+1} := \{A^+_{v|k} \mathcal{P}_k; \quad v \in \mathbb{C}^d\} \quad (5.15)$$

is a vector subspace of $\mathcal{P}_{k+1}]$ with a monic basis of order $k+1$.

PROOF. By assumption \mathcal{P}_k has a monic basis of order k , i. e. a linear basis $(\xi_{k,h})_{h \in F_k}$ which is a perturbation of a monomial basis. Let us first prove that, under our assumptions, $A_{v|k}^+ \xi_k$ are polynomial of degree $k+1$, where ξ_k is a non-zero polynomial of \mathcal{P}_k . By assumption, ξ_k is a polynomial of degree k . Therefore, for any $v \in \mathbb{C}^d \setminus \{0\}$, $X_v \xi_k$ is a polynomial of degree $k+1$. Let $(e_j)_{j \in D}$ be a linear basis of \mathbb{C}^d and denote $A_{e_j|k}^\varepsilon := A_{j|k}^\varepsilon$, where $\varepsilon \in \{+, 0, -\}$. From the definition (5.14) of $A_{v|k}^+$ we know that, for each coordinate function X_j , one has

$$A_{j|k}^+ \xi_k = X_j \xi_k - A_{j|k}^0 \xi_k - A_{j|k}^- \xi_k.$$

The assumptions on $A_{j|k}^0$ and $A_{j|k}^-$ imply that $A_{j|k}^0 \xi_k + A_{j|k}^- \xi_k$ is a polynomial of degree less or equal to k . Therefore, when ξ_k varies in a perturbation of a monomial basis of \mathcal{P}_k and X_j varies among all coordinate functions, $A_{j|k}^+ \xi_{k,h}$ defines a perturbation of a monomial basis of order $k+1$. We define \mathcal{P}_{k+1} to be the linear span of this basis. Then, it is clear that

$$\mathcal{P}_{k+1}] = \mathcal{P}_k] \dot{+} \mathcal{P}_{k+1}$$

where the sums on the right hand side are direct because the non-zero elements of the space \mathcal{P}_{k+1} are polynomials of degree $k+1$. Moreover, \mathcal{P}_{k+1} coincides with the subspace given in (5.15). \blacksquare

Lemma 5.4 *In the notations of Definition 5.1, let be given a 3-diagonal decomposition of $\mathcal{P}_n]$. Define the linear map*

$$a_{v|n}^+ : \mathcal{P}_n \longrightarrow \mathcal{P}_{n+1}] \quad (5.16)$$

by the condition

$$a_{v|n}^+ := X_v \Big|_{\mathcal{P}_n} - a_{v|n}^0 - (a_{v|n-1}^+)^*, \quad v \in \mathbb{C}^d, \quad (5.17)$$

and denote \mathcal{P}_{n+1} the space constructed in Lemma 5.3 with the choices

$$A_{v|k}^0 := a_{v|k}^0 \quad \text{and} \quad A_{v|k}^- := a_{v|k}^- = (a_{v|k-1}^+)^*, \quad k \in \{0, 1, \dots, n\}.$$

The 3-diagonal decompositions of $\mathcal{P}_{n+1}]$ extending the given one are in one to one correspondence with the pairs

$$\left(\langle \cdot, \cdot \rangle_{n+1}, a_{\cdot|n+1}^0 \right) \quad (5.18)$$

where:

(i) $\langle \cdot, \cdot \rangle_{n+1}$ *is a pre-scalar product on \mathcal{P}_{n+1} ,*

(ii) $a_{\cdot|n+1}^0$ *is a linear map*

$$a_{\cdot|n+1}^0 : v \in \mathbb{C}^d \longmapsto a_{v|n+1}^0 \in \mathcal{L}_a(\mathcal{P}_{n+1}) \quad (5.19)$$

such that, for all $v \in \mathbb{R}^d$, $a_{v|n+1}^0$ is a self-adjoint operator on the pre-Hilbert space $(\mathcal{P}_{n+1}, \langle \cdot, \cdot \rangle_{n+1})$.

PROOF. Definition 5.1 implies that any 3-diagonal decompositions of \mathcal{P}_{n+1} extending the given one determines a pair (5.18) with properties (i) and (ii) above.

Conversely, since the linear operators $a_{v|n}^+$ ($v \in \mathbb{C}^d$), hence by Lemma 5.3 the vector space \mathcal{P}_{n+1} , are uniquely determined by condition (5.17), which depends only on the given 3-diagonal decompositions of \mathcal{P}_n , and since the identity (5.5) is satisfied by construction because of (5.17), it follows that any choice of a pair (5.18), satisfying conditions (i) and (ii) above, will define a 3-diagonal decomposition of \mathcal{P}_{n+1} extending the given one. That \mathcal{P}_{n+1} has a monic basis of order $n + 1$ follows from Lemma 5.3. \blacksquare

Theorem 5.5 *The 3-diagonal decompositions of \mathcal{P} are in one-to-one correspondence with the pre-scalar products on \mathcal{P} induced by some state φ on \mathcal{P} .*

PROOF. If the pre-scalar product on \mathcal{P} is induced by a state φ on \mathcal{P} , then by Lemma 2.6 the operators of multiplication by the coordinates are symmetric for this pre-scalar product and the quantum decompositions of the random variables X_j constructed in section 4 provides a 3-diagonal decompositions of \mathcal{P} .

Conversely, let be given a 3-diagonal decompositions of \mathcal{P} of the form (5.10). Then conditions (5.4) and (5.14) imply that for all $v \in \mathbb{R}^d$ one has

$$X_v = \sum_{k \in \mathbb{N}} X_v|_{\mathcal{P}_k} = \sum_{k \in \mathbb{N}} a_{v|k}^+ + \sum_{k \in \mathbb{N}} a_{v|k}^0 + \sum_{k \in \mathbb{N}} (a_{v|k-1}^+)$$

with the convention that $a_{v|-1}^+ = 0$ and where the identity holds on the algebraic linear span of the \mathcal{P}'_k s. Thus, denoting $*$ the adjoint with respect to the pre-scalar product on \mathcal{P}

$$\langle \cdot, \cdot \rangle = \bigoplus_{k \in \mathbb{N}} \langle \cdot, \cdot \rangle_k$$

induced by the 3-diagonal decomposition according to (5.7) and (5.8), one has, on the same domain,

$$\begin{aligned} X_v^* &= \sum_{k \in \mathbb{N}} (a_{v|k}^+)^* + \sum_{k \in \mathbb{N}} (a_{v|k}^0)^* + \sum_{k \in \mathbb{N}} a_{v|k-1}^+ \\ &= \sum_{k \in \mathbb{N}} (a_{v|k}^+)^* + \sum_{k \in \mathbb{N}} a_{v|k}^0 + \sum_{k \in \mathbb{N}} a_{v|k-1}^+ \\ &= \sum_{k \in \mathbb{N}} a_{v|k}^+ + \sum_{k \in \mathbb{N}} a_{v|k}^0 + \sum_{k \in \mathbb{N}} (a_{v|k-1}^+)^* = X_v. \end{aligned}$$

This proves the symmetry of X_v on the given dense domain.

The thesis then follows from Lemma 2.6. \blacksquare

5.2 3-diagonal decompositions of \mathcal{P} and symmetric tensor products

Let \mathcal{P}_{n+1} be the $(n + 1)$ -st space of a 3-diagonal decomposition of \mathcal{P} . From Lemma 2.5 we know that \mathcal{P}_{n+1} is linearly isomorphic to the symmetric tensor power $(\mathbb{C}^d)^{\widehat{\otimes}(n+1)}$. The commutativity of creators (4.14) allows to fix this isomorphism in a canonical way once given a 3-diagonal decomposition of \mathcal{P} .

Lemma 5.6 For $n \in \mathbb{N}^*$, let \mathcal{P}_n be the n -th space of a 3-diagonal decomposition of \mathcal{P} . Denoting, for $v \in \mathbb{C}^d$, $a_v^+ := \sum_{k \in \mathbb{N}} a_{v|k}^+$, the map

$$v_n \widehat{\otimes} v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1 \in (\mathbb{C}^d)^{\widehat{\otimes} n} \longmapsto a_{v_n}^+ a_{v_{n-1}}^+ \cdots a_{v_1}^+ 1_{\mathcal{P}} \in \mathcal{P}_n, \quad n \in \mathbb{N}^*, \quad (5.20)$$

where $\widehat{\otimes}$ denotes symmetric tensor product, extends uniquely to a vector space isomorphism

$$U_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n$$

with the property that for all $v \in \mathbb{C}^d$ and $\xi_{n-1} \in (\mathbb{C}^d)^{\widehat{\otimes}(n-1)}$:

$$U_n(v \widehat{\otimes} \xi_{n-1}) = a_v^+ U_{n-1} \xi_{n-1}. \quad (5.21)$$

Notation. For $n = 0$ we put

$$U_0 : z \in \mathbb{C} := (\mathbb{C}^d)^{\widehat{\otimes} 0} \longmapsto U_0(z) := z \in \mathbb{C} \cdot 1_{\mathcal{P}} \in \mathcal{P}_0. \quad (5.22)$$

PROOF. The map

$$v_n \otimes v_{n-1} \otimes \cdots \otimes v_1 \in (\mathbb{C}^d)^{\otimes n} \longmapsto a_{v_n}^+ a_{v_{n-1}}^+ \cdots a_{v_1}^+ 1_{\mathcal{P}} \in \mathcal{P} \quad (5.23)$$

is well defined. Since both sides in (5.23) are multi-linear, by the universal property of the tensor product it extends to a linear map, denoted $\widehat{\pi}_n$, of $(\mathbb{C}^d)^{\otimes n}$ into \mathcal{P}_n . This map is surjective because by definition

$$\mathcal{P}_n := \{a_{v|n-1}^+(\mathcal{P}_{n-1}); v \in \mathbb{C}^d\} \quad (5.24)$$

and by induction this implies that the vectors of the form

$$a_{v_n}^+ a_{v_{n-1}}^+ \cdots a_{v_1}^+ 1_{\mathcal{P}}, \quad (5.25)$$

with $v_n, v_{n-1}, \dots, v_1 \in \mathbb{C}^d$, are generators of \mathcal{P}_n . Since the right hand side of (5.23) is invariant under permutations of the vectors v_n, v_{n-1}, \dots, v_1 , $\widehat{\pi}_n$ induces a linear map of the vector space of equivalence classes of elements of $(\mathbb{C}^d)^{\otimes n}$ with respect to the equivalence relation induced by the linear action of the permutation group. Since this quotient space is canonically isomorphic to the symmetric tensor product $(\mathbb{C}^d)^{\widehat{\otimes} n}$, this induced map, denoted U_n , defines a linear extension of the map (5.20).

In order to prove that this extension is an isomorphism, we have to prove injectivity. This can be deduced from Lemma 2.5 and the fact that

$$\dim(\mathbb{C}^d)^{\widehat{\otimes} n} = \dim(\mathcal{P}_n) < \infty.$$

■

6 The multi-dimensional Favard Lemma

Theorem 6.1 The 3-diagonal decompositions of \mathcal{P} are in one-to-one correspondence with the pairs of sequences

$$\left((\Omega_n)_{n \in \mathbb{N}}, (\alpha \cdot |n)_{n \in \mathbb{N}} \right)$$

where:

(i) for all $n \in \mathbb{N}$, Ω_n is a linear operator on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ symmetric and positive with respect to the tensor scalar product $\langle \cdot, \cdot \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}}$ defined by (8.9);

(ii) denoting for all $n \in \mathbb{N}$

$$\langle \xi_n, \eta_n \rangle_n := \langle \xi_n, \Omega_n \eta_n \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}}, \quad \xi_n, \eta_n \in (\mathbb{C}^d)^{\widehat{\otimes} n}, \quad (6.1)$$

the pre-scalar product on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ defined by Ω_n and $|\cdot|_n$ the associated pre-norm, one has for all $n \in \mathbb{N}$, $v \in \mathbb{C}^d$ and $\eta_{n-1} \in (\mathbb{C}^d)^{\widehat{\otimes}(n-1)}$:

$$|\eta_{n-1}|_{n-1} = 0 \rightarrow |v \widehat{\otimes} \eta_{n-1}|_n = 0 \quad (6.2)$$

(iii) for all $n \in \mathbb{N}$,

$$\alpha_{\cdot|n} : v \in \mathbb{C}^d \mapsto \alpha_{v|n} \in \mathcal{L}_a((\mathbb{C}^d)^{\widehat{\otimes} n}) \quad (6.3)$$

is a linear map and for all $v \in \mathbb{R}^d$, $\alpha_{v|n}$ is a linear operator on $(\mathbb{C}^d)^{\widehat{\otimes} n}$, symmetric for the pre-scalar product on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ given by (6.1), i.e.,

$$\langle \alpha_{v|n} \xi_n, \Omega_n \eta_n \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}} =: \langle \alpha_{v|n} \xi_n, \eta_n \rangle_n = \langle \xi_n, \alpha_{v|n} \eta_n \rangle_n = \langle \xi_n, \Omega_n \alpha_{v|n} \eta_n \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}} \quad (6.4)$$

(notice that (6.4) implies that $\alpha_{v|n}$ maps the space of $\langle \cdot, \cdot \rangle_n$ -zero norm vectors into itself)

(iv) The sequence Ω_n defines a symmetric interacting Fock space structure over \mathbb{C}^d , endowed with the tensor scalar product, in the sense of Lemma 8.5 (see Appendix 8) and the operator

$$U := \bigoplus_{k \in \mathbb{N}} U_k : \bigoplus_{k \in \mathbb{N}} ((\mathbb{C}^d)^{\widehat{\otimes} k}, \langle \cdot, \cdot \rangle_k) \longrightarrow \bigoplus_{k \in \mathbb{N}} (\mathcal{P}_k, \langle \cdot, \cdot \rangle_{\mathcal{P}_k}) = (\mathcal{P}, \langle \cdot, \cdot \rangle) \quad (6.5)$$

is an orthogonal gradation preserving unitary isomorphism of pre-Hilbert spaces.

PROOF. Given a 3-diagonal decomposition of \mathcal{P} , let $\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n$ respectively $(a_{\cdot|n}^0)_n$ the associated orthogonal gradation of \mathcal{P} and the associated sequence of preservation operators. For $n \in \mathbb{N}$ the linear isomorphism $U_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n$, constructed in Lemma 5.6 is used to transport the pre-scalar product from \mathcal{P}_n to a pre-scalar product $\langle \cdot, \cdot \rangle_n$ on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ through the prescription

$$\langle \xi_n, \eta_n \rangle_n := \langle U_n(\xi_n), U_n(\eta_n) \rangle_{\mathcal{P}_n}. \quad (6.6)$$

so that U_n becomes a unitary isomorphism of pre-Hilbert spaces. Since $(\mathbb{C}^d)^{\widehat{\otimes} n}$ is finite dimensional, the scalar product (6.6) is implemented by a linear operator $\Omega_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow (\mathbb{C}^d)^{\widehat{\otimes} n}$ which is positive and self-adjoint with respect to the tensor scalar product $\langle \cdot, \cdot \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}}$ defined by (8.9):

$$\langle \xi_n, \eta_n \rangle_n := \langle \xi_n, \Omega_n \eta_n \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}}. \quad (6.7)$$

To prove the implication (6.2) notice that, by construction, for any $\eta_{n-1} \in (\mathbb{C}^d)^{\widehat{\otimes}(n-1)}$ one has

$$|\eta_{n-1}|_{n-1} = 0 \iff |U_{n-1} \eta_{n-1}|_{\mathcal{P}_{n-1}} = 0.$$

Using the identity (5.21) and the fact that U_n is a unitary isomorphism one finds

$$|v\widehat{\otimes}\eta_{n-1}|_n = |U_n(v\widehat{\otimes}\eta_{n-1})|_{\mathcal{P}_n} = |a_v^+ U_{n-1}\eta_{n-1}|_{\mathcal{P}_n}$$

and (6.2) follows because, from Lemma 4.7, we know that a_v^+ preserves the ideal of zero norm vectors. Now define

$$\alpha \cdot |n := U_n^{-1} a^0 \cdot |n U_n. \quad (6.8)$$

The map (6.3) is linear and for all $v \in \mathbb{R}^d$, $\alpha_{v|n}$ is a linear operator on $(\mathbb{C}^d)^{\widehat{\otimes} n}$, symmetric for the pre-scalar product (6.1) because the map $a_{v|n}^0$ has these properties with respect to the pre-scalar product $\langle \cdot, \cdot \rangle_{\mathcal{P}_n}$.

Since the orthogonal sum of unitary isomorphism is an orthogonal gradation preserving unitary isomorphism, (iv) follows.

Conversely, given a sequence $((\Omega_n)_{n \in \mathbb{N}}, (\alpha \cdot |n)_{n \in \mathbb{N}})$ satisfying (i), (ii), (iii), (iv) above, one constructs inductively a 3-diagonal decomposition of \mathcal{P} as follows. One starts from the identification

$$\mathcal{P}_{0]} = \mathbb{C} \cdot 1_{\mathcal{P}} \equiv \mathbb{C} \equiv (\mathbb{C}^d)^{\widehat{\otimes} 0}$$

where the choice of the scalar product is induced by Ω_0 . Having defined a 3-diagonal decomposition of $\mathcal{P}_{n]} = \bigoplus_{k=0}^n \mathcal{P}_k$, one constructs the decomposition

$$\mathcal{P}_{n]} = \mathcal{P}_{n-1]} \dot{+} \mathcal{P}_n$$

as in Lemma 5.4 and from Lemma 5.6 one has the identification $U_n : (\mathbb{C}^d)^{\widehat{\otimes} n} \rightarrow \mathcal{P}_n$. Using U_n one defines $a^0 \cdot |n := U_n \alpha \cdot |n U_n^{-1}$ and this allows to define, as in Lemma 5.4, a vector subspace $\mathcal{P}_{n+1} \subset \mathcal{P}_{n+1]}$, with a monic basis of order $n+1$ thus in particular

$$\mathcal{P}_{n+1]} = \mathcal{P}_{n+1]} \dot{+} \mathcal{P}_{n+1}.$$

Using Lemma 5.6 \mathcal{P}_{n+1} is identified, as vector space, with $(\mathbb{C}^d)^{\widehat{\otimes}(n+1)}$ and this allows to transport the Ω_{n+1} -pre-scalar product on \mathcal{P}_{n+1} . Using this pre-scalar product, we define $a^{\cdot |n+1}$ and this allows to iterate the construction of the 3-diagonal decomposition of \mathcal{P} . ■

Theorem 6.2 *Let μ be a probability measure on \mathbb{R}^d with finite moments of all orders and denote φ the state on \mathcal{P} given by*

$$\varphi(b) := \int_{\mathbb{R}^d} b(x_1, \dots, x_d) d\mu(x_1, \dots, x_d), \quad b \in \mathcal{P} \quad (6.9)$$

Then there exist two sequences

$$(\Omega_n)_{n \in \mathbb{N}}, \quad (\alpha \cdot |n)_{n \in \mathbb{N}}$$

satisfying conditions (i), (ii), (iii), (iv) of Theorem 6.1. Moreover, denoting

$$\Gamma(\mathbb{C}^d; (\Omega_n)_n) := \bigoplus_{n \in \mathbb{N}} \left((\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}} \right) \quad (6.10)$$

the symmetric interacting Fock pre-Hilbert space defined by the sequence $(\Omega_n)_{n \in \mathbb{N}}$, A^{\pm} the creation and annihilation fields associated to it, $P_{\Gamma, n}$ the projection onto the n -th

space of the gradation (6.10), and N the number operator associated to this gradation i.e.

$$N := \sum_{n \in \mathbb{N}} n P_{\Gamma, n}$$

the gradation preserving unitary pre-Hilbert space isomorphism defined by (6.5) satisfies

$$U\Phi = 1_{\mathcal{P}} \quad (6.11)$$

$$U^{-1}X_v U = A_v^+ + \alpha_{v,N} + A_v^-, \quad \forall v \in \mathbb{R}^d \quad (6.12)$$

where $\alpha_{v,N}$ is the symmetric operator defined by:

$$\alpha_{v,N} := \sum_{n \in \mathbb{N}} \alpha_{v|n} P_{\Gamma, n}$$

Conversely, given two sequences (Ω_n) and $(\alpha_{\cdot|n})$ satisfying conditions (i), (ii), (iii), (iv) of Theorem 6.1, there exists a state φ on \mathcal{P} , induced by a probability measure on \mathbb{R}^d in the sense of Lemma 2.6, such that for any probability measure μ on \mathbb{R}^d , inducing the state φ on \mathcal{P} , the pair of sequences $(\Omega_n)_{n \in \mathbb{N}}, (\alpha_{\cdot|n})_{n \in \mathbb{N}}$ is the one associated to μ according to the first part of the theorem.

PROOF. The prescription (6.9) establishes a one-to-one correspondence between moment equivalence classes of probability measures on \mathbb{R}^d with moments of all order and states φ on \mathcal{P} satisfying the conditions of Lemma 2.6.

By Theorem 5.5 the states φ on \mathcal{P} with this property are in one-to-one correspondence with the 3-diagonal decompositions of \mathcal{P} and Theorem 6.1 establishes a one-to-one correspondence between 3-diagonal decompositions of \mathcal{P} and pairs of sequences $(\Omega_n)_{n \in \mathbb{N}}, (\alpha_{\cdot|n})_{n \in \mathbb{N}}$ with the properties stated in the theorem.

The constructive form of this correspondence given in Lemma 5.3 shows that the identity (5.17) holds and, through the definitions (6.7) and (6.8), this is equivalent to (6.12). ■

7 Appendix: Orthogonal projectors on pre-Hilbert spaces

Definition 7.1 We use the following terminology:

- (1) A pre-scalar product on a vector space V is a positive definite Hermitean form on V .
- (2) A scalar product on a vector space V is a non-degenerate pre-scalar product on V .
- (3) A pre-Hilbert space is a vector space equipped with a pre-scalar product.
- (4) A Hilbert space is a vector space equipped with a scalar product and complete with respect to the topology induced by it.

Given two pre-Hilbert spaces \mathcal{H} and \mathcal{K} , we denote $\mathcal{L}_a(\mathcal{H}, \mathcal{K})$ the space of all adjointable linear operators from \mathcal{H} to \mathcal{K} . This means that, if A is such an operator, then

$$\langle Ah, k \rangle_{\mathcal{K}} = \langle h, A^*k \rangle_{\mathcal{H}},$$

in particular A is everywhere defined on \mathcal{H} .

Lemma 7.2 *Let \mathcal{K} be a separable \mathbb{C} -pre-Hilbert space with pre-scalar product $(\cdot, \cdot)_{\mathcal{K}}$ and let*

$$(k_j)_{j \in D_0 \cup D_1}$$

be a linear basis of \mathcal{K} such that $(k_j)_{j \in D_0}$ is a linear basis of the subspace

$$\mathcal{K}_0 := \{v \in \mathcal{K} ; |v|_{\mathcal{K}} = 0\}.$$

(i) *Then for every numeration of D_1 , i.e., an identification*

$$D_1 \equiv \{1, 2, \dots, d \leq \infty\},$$

*there exists an orthonormal set $(e_j)_{j \in D_1}$ of \mathcal{K} with the following property:
for all $m \in \{1, 2, \dots, d \leq \infty\}$*

$$\text{lin-span} \{k_j ; j \in \{1, \dots, m\}\} = \text{lin-span} \{e_j ; j \in \{1, \dots, m\}\}. \quad (7.1)$$

In particular the set

$$(e_j)_{j \in D_1} \cup (k_j)_{j \in D_0}$$

is a linear basis of \mathcal{K} .

(ii) *If the $(\cdot, \cdot)_{\mathcal{K}}$ -scalar products of the k_j are in \mathbb{R} , then property (7.1) holds also for the real linear span.*

PROOF. Define

$$e_1 := k_1 / |k_1|_{\mathcal{K}}.$$

Having defined an orthonormal set $(e_j)_{j \in \{1, \dots, n \leq d\}}$ satisfying (7.1), for each $m \leq n$, define

$$e_{n+1}^0 := k_{n+1} - \sum_{i=1}^n (e_i, k_{n+1}) e_i. \quad (7.2)$$

$|e_{n+1}^0|_{\mathcal{K}} \neq 0$ because of the linear independence of the k_j 's. Define

$$e_{n+1} := e_{n+1}^0 / |e_{n+1}^0|_{\mathcal{K}}.$$

By construction:

$$\text{lin-span}\{e_m ; m \in \{1, \dots, n+1\}\} = \text{lin-span}\{k_m ; m \in \{1, \dots, n+1\}\}$$

$$(e_j, e_i)_{\mathcal{K}} = \delta_{ij}, \quad \forall i, j \in \{1, \dots, n+1\}.$$

Thus the set $\{e_1, \dots, e_{n+1}\}$ is orthonormal. Therefore by induction one obtains a sequence $(e_i)_{i \in D_1}$ with the required properties. Finally (ii) follows from (7.2). \blacksquare

Corollary 7.3 *Let \mathcal{K} be a separable \mathbb{C} -pre-Hilbert space with pre-scalar product $(\cdot, \cdot)_{\mathcal{K}}$ and let $\mathcal{K}_0, \mathcal{K}_1$ be sub-spaces of \mathcal{K} such that \mathcal{K}_1 is a linear supplement of \mathcal{K}_0 in \mathcal{K} , i.e.*

$$\mathcal{K} = \mathcal{K}_0 \dot{+} \mathcal{K}_1. \quad (7.3)$$

Denote $\mathcal{K}_{0,0}$ the zero norm subspace of \mathcal{K}_0 and let $\mathcal{K}_{0,1}$ be any linear supplement of $\mathcal{K}_{0,0}$ in \mathcal{K}_0

$$\mathcal{K}_0 = \mathcal{K}_{0,0} \dot{+} \mathcal{K}_{0,1}. \quad (7.4)$$

Then there exists a unique self-adjoint projection $P_{\mathcal{K}_0}$ from \mathcal{K} onto \mathcal{K}_0 such that, for each $k_1 \in \mathcal{K}_1$, $P_{\mathcal{K}_0}(k_1) \in \mathcal{K}_{0,1}$ and $P_{\mathcal{K}_0}(k_1)$ has zero-norm if and only if k_1 is orthogonal to \mathcal{K}_0 .

PROOF. From Lemma 7.2 we know that there exists a linear basis $(e_j)_{j \in D_1 \cup D_0}$ of \mathcal{K}_0 such that $(e_j)_{j \in D_1}$ is an orthonormal basis of $\mathcal{K}_{0,1}$ and $(e_j)_{j \in D_0}$ is a linear basis of $\mathcal{K}_{0,0}$. The linear operator

$$P_{\mathcal{K}_0} : k_0 + k_1 \in \mathcal{K}_0 + \mathcal{K}_1 \longmapsto P_{\mathcal{K}_0}(k_0 + k_1) := k_0 + \sum_{j \in D_1 \cup D_0} (e_j, k_1)_k e_j = k_0 + \sum_{j \in D_1} (e_j, k_1)_k e_j$$

is well defined because, for any $k \in \mathcal{K}$, the decomposition

$$k = k_0 + k_1, \quad k_0 \in \mathcal{K}_0, \quad k_1 \in \mathcal{K}_1$$

is unique. Notice that $P_{\mathcal{K}_0}(k_1) \in \mathcal{K}_{0,1}$. Since the e_j 's with $j \in D_1$ are an orthonormal set, then $k_1 \in \mathcal{K}_1$ is such that $P_{\mathcal{K}_0}(k_1)$ has zero norm if and only if one has

$$(e_j, k_1)_k = 0, \quad \forall j \in D_1$$

and, since $(e_j, k_1) = 0$ for all $j \in D_0$ it follows that k_1 is orthogonal to \mathcal{K}_0 . The self-adjointness of $P_{\mathcal{K}_0}$ follows from a direct calculation. The uniqueness of the operator with the required properties, follows from self-adjointness and the fact that $P_{\mathcal{K}_0}(\mathcal{K}_1) \subseteq \mathcal{K}_{0,1}$. \blacksquare

Definition 7.4 *In the notations of Corollary 7.3, $P_{\mathcal{K}_0}$ will be called the orthogonal projection onto \mathcal{K}_0 associated to the decompositions (7.3) and (7.4).*

8 Appendix: Symmetric interacting Fock spaces

Definition 8.1 *Let V be a vector space and denote, for all $n \in \mathbb{N}$, $V^{\widehat{\otimes} n}$ the n -th symmetric algebraic tensor power of V , where by definition*

$$V^{\widehat{\otimes} 0} := \mathbb{C} \cdot \Phi, \quad \langle \Phi, \Phi \rangle_0 = 1. \quad (8.1)$$

A symmetric interacting Fock space (IFS) structure over V is a sequence $(\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}$ such that:

- (i) *for all $n \in \mathbb{N}$, $\langle \cdot, \cdot \rangle_n$ is a pre-scalar product on $V^{\widehat{\otimes} n}$,*
- (ii) *for all $v \in V$ and for all $n \in \mathbb{N}$, the map*

$$A_{v|n}^+ : \xi_n \in V^{\widehat{\otimes} n} \longrightarrow v \widehat{\otimes} \xi_n \in V^{\widehat{\otimes}(n+1)} \quad (8.2)$$

which at algebraic level is always well defined and is called the symmetric creator of order n with test function v , has an adjoint as a pre-Hilbert space operator (thus in particular everywhere defined on $V^{\widehat{\otimes} n}$)

$$A_{v|n}^- : (V^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_n) \longrightarrow (V^{\widehat{\otimes}(n-1)}, \langle \cdot, \cdot \rangle_{n-1}).$$

In this case one says that the sequence of scalar products $(\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}$ satisfies the symmetric IFS-compatibility condition.

Remark. The existence of the pre-Hilbert space adjoint of the symmetric creator of order n with test function v is equivalent to say that, for each $v \in V$ and each $n \in \mathbb{N} \setminus \{0\}$, there exists a map

$$A_{v|n}^- : V^{\widehat{\otimes} n} \longrightarrow V^{\widehat{\otimes}(n-1)} \quad (8.3)$$

such that

$$\langle \xi_n, v \widehat{\otimes} \eta_{n-1} \rangle_n = \langle A_{v|n}^- \xi_n, \eta_{n-1} \rangle_{n-1}, \quad \forall \xi_n \in V^{\widehat{\otimes} n}, \forall \eta_{n-1} \in V^{\widehat{\otimes}(n-1)}. \quad (8.4)$$

The above prescription defines $A_{v|n}^-$ only for $n \geq 1$. Defining for all $v \in V$

$$A_v^- := \sum_{n \geq 1} A_{v|n}^-, \quad A_v^+ := \sum_{n \geq 0} A_{v|n}^+,$$

the only extension of A_v^- to $V^{\widehat{\otimes} 0} := \mathbb{C} \cdot \Phi$ compatible with the identity

$$(A_v^-)^* \Big|_{V^{\widehat{\otimes} n}} = A_v^+ \Big|_{V^{\widehat{\otimes}(n-1)}}, \quad \forall n \in \mathbb{N}$$

is the Fock prescription

$$V^{\widehat{\otimes}(-1)} = \{0\} \Leftrightarrow A_{v|0}^- \Phi = 0, \quad \forall v \in V. \quad (8.5)$$

In the following we will assume the validity of (8.5).

Remark. The identity (8.4) implies in particular that a necessary condition for the existence of the pre-Hilbert space adjoint of $A_{v|n}^+$ for any $v \in V$ is that denoting, for all $n \in \mathbb{N}$, $|\cdot|_n$ the norm on $V^{\widehat{\otimes} n}$ induced by the pre-scalar product $\langle \cdot, \cdot \rangle_n$, one has:

$$\forall \eta_{n-1} \in V^{\widehat{\otimes}(n-1)}, \quad |\eta_{n-1}|_n = 0 \implies |A_{v|n-1}^+ \eta_{n-1}|_n = 0, \quad \forall v \in V, \forall n \in \mathbb{N}. \quad (8.6)$$

i.e. that for any $v \in V$ $A_{v|n-1}^+$ maps the zero space of $\langle \cdot, \cdot \rangle_{n-1}$ into the zero space of $\langle \cdot, \cdot \rangle_n$. If V is finite-dimensional the condition is also sufficient and (8.6) is equivalent to (8.4).

Remark. We denote, for each $n \in \mathbb{N}$, $\mathcal{H}_n(V)$ the completion of the quotient $V^{\widehat{\otimes} n}$ modulo the $\langle \cdot, \cdot \rangle_n$ -zero norm vectors and, when no confusion is possible, we denote the scalar product on $\mathcal{H}_n(V)$ with the same symbol $\langle \cdot, \cdot \rangle_n$. With these notations, on the vector space direct sum

$$\bigoplus_{n \in \mathbb{N}} \mathcal{H}_n(V) \quad (8.7)$$

i.e. the space of sequences $(\xi_n)_{n \in \mathbb{N}}$ with $\xi_n \in \mathcal{H}_n(V)$ for all $n \in \mathbb{N}$ and $\xi_n = 0$ for almost all $n \in \mathbb{N}$, there is a unique scalar product, denoted $\langle \cdot, \cdot \rangle$, with the property that the spaces $\mathcal{H}_n(V)$ are mutually orthogonal and

$$\langle \xi_n, \eta_n \rangle := \langle \xi_n, \eta_n \rangle_n, \quad \forall \xi_n, \eta_n \in \mathcal{H}_n(V), \quad \forall n \in \mathbb{N},$$

where, here and in the following, an element $\xi_n \in \mathcal{H}_n(V)$ ($n \in \mathbb{N}$) is identified to the sequence $(\xi_k)_{k \in \mathbb{N}}$ with $\xi_k = 0$ for $k \neq n$. The completion of the vector space (8.7) for the scalar product $\langle \cdot, \cdot \rangle$ consists of all the sequences $(\xi_n)_{n \in \mathbb{N}}$ with $\xi_n \in \mathcal{H}_n(V)$ and

$$\sum_{n \in \mathbb{N}} \langle \xi_n, \xi_n \rangle_n < \infty$$

and will be denoted

$$\Gamma(V; (\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}). \quad (8.8)$$

Definition 8.2 The Hilbert space (8.8) will be called the symmetric interacting Fock space over V with defining sequence $(\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}$. In the following we will use the notation

$$\Gamma(V; (\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}) = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n(V).$$

Lemma 8.3 Let $\Gamma(V; (\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}})$ be an IFS over V and for all $n \in \mathbb{N}$ denote π_n the canonical projection of $V^{\widehat{\otimes} n}$ onto the quotient $V^{\widehat{\otimes} n} / \langle \cdot, \cdot \rangle_n$ and consider the canonical embedding $j_{0,n} : V^{\widehat{\otimes} n} / \langle \cdot, \cdot \rangle_n \rightarrow \mathcal{H}_n(V)$. Define $j_n : V^{\widehat{\otimes} n} \rightarrow \mathcal{H}_n(V)$ by $j_n := j_{0,n} \circ \pi_n$. Then, for all $v \in V$, the linear maps

$$\begin{aligned} a_{v|n}^+ : j_n(\xi_n) \in j_n(V^{\widehat{\otimes} n}) &\longmapsto j_{n+1}(v \widehat{\otimes} \xi_n) \in j_{n+1}(V^{\widehat{\otimes}(n+1)}) \\ a_{v|n}^- : j_n(\xi_n) \in j_n(V^{\widehat{\otimes} n}) &\longmapsto j_{n-1}(A_{v|n}^- \xi_n) \in j_{n-1}(V^{\widehat{\otimes}(n-1)}) \end{aligned}$$

where by definition

$$V^{\widehat{\otimes}(-1)} = \{0\},$$

are mutually adjoint on their domain, i.e.

$$\langle a_{v|n}^+ j_n(\xi_n), j_{n+1}(\eta_{n+1}) \rangle_{n+1} = \langle j_n(\xi_n), a_{v|n}^- j_{n+1}(\eta_{n+1}) \rangle_n.$$

PROOF. Let $\xi_n \in V^{\widehat{\otimes} n}$ and $\gamma_{n+1} \in V^{\widehat{\otimes}(n+1)}$. Then, one has

$$\begin{aligned} \langle a_{v|n}^+ j_n(\xi_n), j_{n+1}(\gamma_{n+1}) \rangle_{n+1} &= \langle j_{n+1}(v \widehat{\otimes} \xi_n), j_{n+1}(\gamma_{n+1}) \rangle_{n+1} \\ &= \langle v \widehat{\otimes} \xi_n, \gamma_{n+1} \rangle_{n+1} = \langle \xi_n, A_{v|n+1}^- \gamma_{n+1} \rangle_n \\ &= \langle j_n(\xi_n), j_n(A_{v|n+1}^- \gamma_{n+1}) \rangle_n = \langle j_n(\xi_n), a_{v|n+1}^- \gamma_{n+1} \rangle_n. \end{aligned}$$

This proves

$$a_{v|n}^- = (a_{v|n-1}^+)^*.$$

■

Definition 8.4 In the notations of Definition 8.1, suppose that V is a pre-Hilbert space with pre-scalar product $\langle \cdot, \cdot \rangle_V$. Then for each $n \in \mathbb{N}$, the algebraic tensor product $V^{\widehat{\otimes} n}$ is a pre-Hilbert space with the pre-scalar product uniquely determined by the condition

$$\langle u^{\widehat{\otimes} m}, v^{\widehat{\otimes} n} \rangle_{V^{\widehat{\otimes} n}} := \delta_{m,n} \langle u, v \rangle_V^n, \quad \forall u, v \in V, \forall n \in \mathbb{N}. \quad (8.9)$$

If, for all $n \in \mathbb{N}$, there exist a symmetric positive operator

$$\Omega_n \in \mathcal{L}_a(V^{\widehat{\otimes} n})$$

such that

$$\langle \xi_n, \eta_n \rangle_n = \langle \xi_n, \Omega_n \eta_n \rangle_{V^{\widehat{\otimes} n}}, \quad \forall \xi_n, \eta_n \in V^{\widehat{\otimes} n}, \quad (8.10)$$

then the symmetric IFS structure on V defined by the sequence $(\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}$ will be called standard and for the associated IFS we will use the notation

$$\Gamma(V; (\Omega_n)).$$

The following Lemma shows that, if V is finite dimensional, then every IFS over V is standard and gives a simple rule to construct such spaces.

Lemma 8.5 *The assignment of a symmetric interacting Fock space structure over \mathbb{C}^d :*

$$\Gamma(\mathbb{C}^d; (\langle \cdot, \cdot \rangle_n)_{n \in \mathbb{N}}) := \bigoplus_{n \in \mathbb{N}} \left((\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_n \right)$$

is equivalent to the assignment of a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of linear operators on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ with the following properties:

- (i) *for all $n \in \mathbb{N}$, Ω_n is positive symmetric with respect to the n -th symmetric tensor power of the Euclidean scalar product on $(\mathbb{C}^d)^{\widehat{\otimes} n}$, denoted $\langle \cdot, \cdot \rangle_{(\mathbb{C}^d)^{\widehat{\otimes} n}}$ in the following;*
- (ii) *for all $n \in \mathbb{N}$ and for all $\xi_n \in (\mathbb{C}^d)^{\widehat{\otimes} n}$*

$$\Omega_n \xi_n = 0 \implies \Omega_{n+1} v \widehat{\otimes} \xi_n = 0, \quad \forall v \in \mathbb{C}^d; \quad (8.11)$$

- (iii) *for all $n \in \mathbb{N}$ the identity (8.10) holds with $V = \mathbb{C}^d$.*

PROOF. The existence of Ω_n is clear since in a finite dimensional space any Hermitian form is continuous. Positivity and symmetry follow from the corresponding properties of $\langle \cdot, \cdot \rangle_n$. Condition (8.11) is equivalent to (8.6). ■

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